2.1 Quadratic Functions: Exploring Forms

Student Text Pages
64–75

Suggested Timing
75 min

Materials and Technology Tools
- grid paper
- graphing calculators
- computers with graphing software (optional)
- algebra tiles (optional)
- linking cubes or colour tiles (optional)

Related Resources
- BLM G-1 Grid Paper
- BLM 2-3 Section 2.1 Quadratic Functions: Exploring Forms

Teaching Suggestions

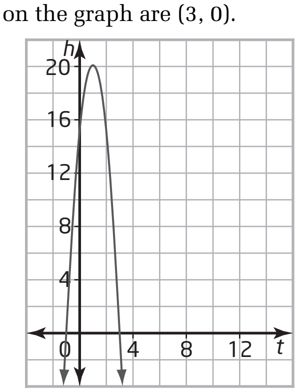
- If you can collaborate with the physics teacher, students could build and test a bottle rockets as part of their physics program. The Making Connections box beside the section opener can help connect mathematical concepts to basic physics principles.

Investigate

- Use graphing software as an alternative to a graphing calculator.
- Students may work individually or in pairs to explore the different forms of the same quadratic equation.
- Ensure that students understand that each form offers unique information about the quadratic function, and thus the motion of the bottle rocket.
- To relate this topic to physics, you may wish to mention that the formula $h(t) = 5t^2 + 10t + 15$ comes from the kinematics formula, $p(t) - p_0 + v_0t + \frac{1}{2}at^2$ where p is the position, v_0 is the initial velocity, t is the time, and a is acceleration. Students will learn more about such formulas in their physics class. In this Investigate, and elsewhere in the chapter, the acceleration due to gravity, 9.81 m/s², has been rounded to 10 m/s², so as to provide factorable polynomials in a real-life context.
- Use **question 6** to emphasize that a mathematical model can extend beyond points for which it has real meaning.

Investigate Responses (pages 64-65)

1. a) The launch platform is 15-m high.
b) The coordinates of the point (0, 15) tell us that at the time of 0 s, the height is 15 m.
2. a) The rocket reaches a maximum height of 20 m at a time of approximately 1 s.
b) The equation of the axis of symmetry is $t = 1$.
c) i) The height of the bottle rocket is increasing for $0 < t < 1$.
ii) The height of the bottle rocket is decreasing for $1 < t < 3$.
d) The maximum height reached is the highest point on the graph. We get the height and time from the coordinates of this point. For all times up to the maximum, the height must be increasing. For all times after the maximum point until landing, the height must be decreasing.
3. a) The rocket was in the air for a total of 3 s. This is the hang time.
b) For all points up until 3 s, we have a positive height, which means the rocket is in the air. When we see a height of 0 m, we know that the rocket has landed. The coordinates of the point on the graph are (3, 0).
4. a) All three graphs for the functions i), ii), and iii) are identical.



b) The three graphs in part a) all contain the same information that is found on the height versus time graph for the bottle rocket.

c) The three functions are different forms of the same function and any of them may be used to express the height–time relationship shown in the graph.

5. a) The function $h(t) = -5(t - 1)^2 + 20$ identifies the coordinates of the vertex. The time at 1 s corresponds to the maximum possible us height of 20 m.

b) The function $h(t) = -5t^2 + 10t + 15$ identifies the h-intercept. The time at 0 s corresponds to the initial height of 15 m.

c) The function $h(t) = -5(t - 3)(t + 1)$ identifies the t-intercepts. The height of 0 m corresponds to two different times, one at 3 s and the other at -1 s.

6. a) The domain of the function is the set of all real numbers.

b) The range of the function is the set of all real numbers less than or equal to 20.

c) The domain for the flight is $0 < t < 3$ and the range for the flight is $0 < h < 20$. So, negative real numbers and real numbers greater than 3 for the domain, and negative real numbers and real numbers greater than 20 for the range will not provide information about the flight of the rocket.

7. a) There are three algebraic forms for expressing the same quadratic equation. The structure of each form is distinct but each can be converted algebraically to either of the other two forms, so any one function or its graph may be expressed in all three of the different forms.

b) Each of the forms has advantages and disadvantages. The standard form is compact and gives the initial value for the function. In the case of the bottle rocket, the standard form gives the initial height of the rocket. The factored form of a quadratic function identifies the horizontal intercepts of the function. In the case of the bottle rocket, the factored form identifies the time at which the rocket returns to the ground. The vertex form of a quadratic function gives the maximum or minimum value of the function and the input value that corresponds to the maximum or minimum. In the case of the bottle rocket, the vertex form identifies the maximum height reached and the time at which it occurred.

Examples

- Time may not permit working through all Examples in detail. Have students read through the Examples independently or in small groups before reviewing them as a class.
- Ask students to pay attention to the key features and the information provided by the three common forms of a quadratic function.
- Encourage students to keep notes on how the three forms are different and to list the advantages and disadvantages of each form. They could also use a table to organize the information.
- Ensure that student have made note of these features of each form of a quadratic function.

  Example 1: Standard Form $y = ax^2 + bx + c$
  - It has terms written in decreasing powers of x.
  - It identifies the direction of opening and the y-intercept of its graph.
  - It is a relatively cumbersome form to sketch a graph without technology.

  Example 2: Factored Form $y = a(x - r)(x - s)$
  - It identifies the direction of opening and the x-intercepts (the zeros of the function) of its graph.
  - The x-intercepts are the constants r and s in the binomial factors.
  - The axis of symmetry is a vertical line midway between the x-intercepts.

  Example 3: Vertex Form $y = a(x - h)^2 + k$
  - It identifies the direction of opening and the vertex (h, k) of its graph.
  - The axis of symmetry is at $x = h$.

Communicate Your Understanding

- For **questions C1 and C4**, ask students to write their answers in their journal or notebook to communicate their understanding in their own words.
- For **questions C2 and C3**, consider using a think-pair-share strategy to assess the understanding of the whole class.
- You may wish to use **BLM 2–3 Section 2.1 Quadratic Functions: Exploring Forms** for remediation or extra practice.

Communicate Your Understanding Responses (page 70)

C1 There are three algebraic forms for expressing quadratic functions. Each form has its advantages and disadvantages. The table shows how to identify key information from all three forms. The vertex form is useful when the coordinates of the vertex or the equation of the axis of symmetry are needed. The standard form is useful when the y-intercept is needed. The factored form is useful when the x-intercepts are needed.

	Standard Form $f(x) = ax^2 + bx + c$	Vertex Form $f(x) = a(x - h)^2 + k$	Factored Form $f(x) = a(x - r)(x - s)$
Direction of opening	Upward for $a > 0$ Downward for $a < 0$	Upward for $a > 0$ Downward for $a < 0$	Upward for $a > 0$ Downward for $a < 0$
Vertex	–	(h, k)	–
Axis of symmetry	–	$x = h$	$x = \frac{r + s}{2}$
Vertical intercept	$(0, c)$	$(0, ah^2 + k)$	$(0, ars)$
Horizontal intercepts	–	–	$x = r, x = s$

C2 **a)** You can immediately identify the y-intercept in the function $h(x) = -3x^2 - 4x + 6$. When $x = 0$, $y = 6$. The y-intercept is 6.

b) For the other two functions, substitute $x = 0$ into the expression and evaluate to find the y-intercept.

For $g(x) = (x - 4)(x - 1)$:
$g(0) = (0 - 4)(0 - 1)$
$= -4$
The y-intercept is -4.

For $f(x) = -2(x - 7)^2 + 3$:
$f(0) = -2(0 - 7)^2 + 3$
$= -95$
The y-intercept is -95.

c) The parabola for the function in ii) opens upward. This is because the expanded form of the function has a positive coefficient on the first term. The parabola for the functions in i) and iii) open downward. This is because the values of a in these functions are negative.

C3 First plot the two given points for the vertex and the y-intercept. Since the horizontal distance, h, from the vertex to the y-intercept is known, you can find the coordinates of the mirror point for $(0, c)$ at $(2h, c)$, which is at the same horizontal distance from the vertex. At this point, if only a rough sketch is needed, use freehand drawing to sketch the parabola that passes through the three points. If a more accurate graph is needed, then calculate the value of a based on $ah^2 + k = c$ from the table: $a = \frac{c - k}{h^2}$. Now the vertex form of the equation, $y = a(x - h)^2 + k$, is known. Use it to generate more points to graph the function.

C4 **a)** To identify the intervals or the values of x for which a quadratic function is positive or negative, first find the x-intercepts of the graph, if any. For a function that opens upward, there are three possible cases. In Case 1, the parabola opens upward with the vertex above the x-axis. In this case, the function is positive for $x \in \mathbf{R}$. In Case 2, the vertex is on the x-axis at $(h, 0)$. In this case, there are two intervals, $x < h$ and $x > h$. In Case 3, the parabola opens upward with the vertex below the x-axis. In this case, the function is positive when $x < r$ and $x > s$. The function is negative between the

Common Errors

- Students may miss one of the zeros of a quadratic function. For example, $f(x) = x(x - 2)$ may appear to have only one binomial factor, $x - 2$, that gives a zero or x-intercept of 2.

R_x Use this strategy to help students see that there are two binomial factors.
$x = x - 0$
$f(x) = x(x - 2)$
$f(x) = (x - 0)(x - 2)$
So, 0 is a zero of a quadratic function with x as a factor.

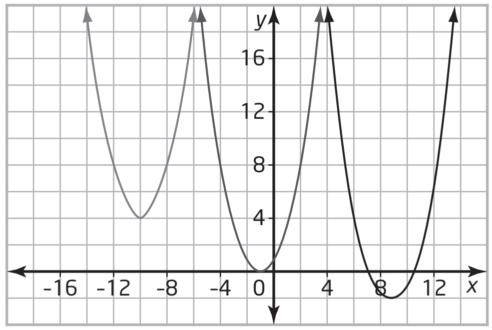
- Some students may have difficulty writing the correct signs for the zeros of a quadratic function from its factored form.

R_x Have students solve for the x-intercepts by setting each factor equal to zero and solving the linear equation.

- Some students may have difficulty writing the correct signs for the coordinates of the vertex of a quadratic function from its vertex form.

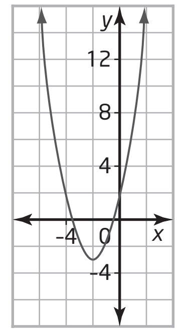
R_x Have students review horizontal and vertical transformations on the graph of $y = x^2$ to draw connections between the graphical and algebraic effects. Consider using graphing technology to expedite this review.

x-intercepts at $r < x < s$. A similar way of identifying the intervals applies to parabolas that open downward.

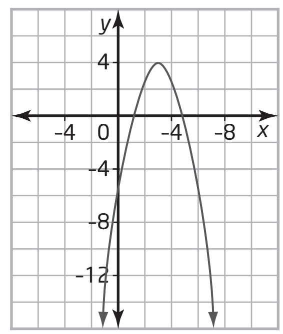


b) To find out for which values of x a function is increasing or decreasing, consider how the y-values of the function change as the values for x move from left to right.

For parabolas that open upward, the function decreases for all values of x to the left of the vertex and increases for all values of x to the right of the vertex. The function $y = (x + 2)^2 - 3$ is decreasing for $x < -2$ and increasing for $x > -2$.



For parabolas that open downward, the function increases for all values of x to the left of the vertex and decreases for all values of x to the right of the vertex. The function $y = -(x - 3)^2 + 4$ is increasing for $x < 3$ and decreasing for $x > 3$.



Practise, Connect and Apply, Extend

- Students should answer **questions 3 to 7** without the use of technology. However, they may be encouraged to use graphing technology to check their work.
- In part d) of **question 4**, it appears that there is only one binomial factor, $x + 6$, in the quadratic function that gives a zero, or x-intercept, of -6. Students may miss the other zero of the function. Remind them that $x = x - 0$, so 0 is another zero of the function.
- For **questions 8 and 9**, students should see that even though the domain of a mathematical model must be restricted for the equation to have physical meaning, the mathematical information, such as the negative x-intercept of the graph, can be useful in developing the equation of the quadratic function. Point out to students that mathematics is often considered primarily a tool (but a powerful one) for physical sciences.
- Part d) of **question 9** provides an opportunity to see if students can make connections between mathematics and the physical situation it represents.
- For **question 10**, students may have difficulty generalizing a process without specific (numerical) examples to start with. If this happens, provide students with several pairs of numbers as x-intercepts of quadratic functions, and ask them to find the x-coordinates of the vertices before they describe a process.

Accommodations

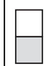
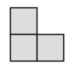
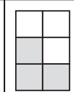
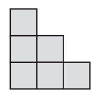
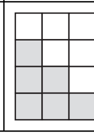
Visual–provide students with colour tiles to build the growing pattern in **question 14**

Motor–have students use technology for graphing

Student Success

- Throughout this chapter, quadratic functions are given in different forms. Ensure that students are familiar with each form and with the information each form provides about the key features of the function and its graph.
- Encourage students to write on index cards or in their notebooks the different forms of a quadratic function and the information each form provides for later reference or review.

- In **question 13**, it is interesting to note how one form collapses into another form when some terms in the quadratic function are zero. This could provoke lively discussion. You may wish to use algebra tiles for part b).
- For **questions 14 to 17**, linking cubes or colour tiles are recommended. For **question 17**, some students may have difficulty arriving at the equation. Use the following visual method to help them.

Strips, *n*	Area Mowed	Double the Area to Form a Rectangle
1		
2		
3		

After n strips, the number of squares is one half the number of squares in a rectangle with width n squares and length $(n + 1)$ squares. Hence, Area mowed $= \frac{1}{2}n(n + 1)$.

Career Connections

- Draw students' attention to the Career Connection on student text page 75. Students who are interested in the architect career described might also be interested in the following related careers:
  - architectural designer
  - architectural technician
  - structural engineer
  - landscape architect
  - construction manager
  - urban planner
- Ask students to choose one of the related careers that interest them most. Research the qualification and skills required for the job, such as university/college education or training.
- You may also ask students to list problems that they might have to solve or difficulties they expect to overcome for the job.

Mathematical Processes Integration

The table shows questions that provide good opportunities for students to use the mathematical processes.

Process Expectations	Selected Questions
Problem Solving	9, 11, 12, 14, 17, 20
Reasoning and Proving	9–21
Reflecting	9, 11, 12, 14, 17–21
Selecting Tools and Computational Strategies	8, 9, 12, 14, 17
Connecting	8, 12, 14, 17, 21
Representing	3, 5, 7, 9–11, 13, 14, 16–20
Communicating	1, 2, 4–7, 9–21