6.3 Investigate Rational Exponents

Student Text Pages
296–304

Suggested Timing
60–70 min

Materials and Technology Tools
- scientific or graphing calculators

Related Resources
- BLM 6–6 Section 6.3 Investigate Rational Exponents

Teaching Suggestions

- The intent of the Investigate is to provide a rationale for the definition of rational exponents. As you work through the lesson, including the Examples, revisit and emphasize the use of the exponent laws in reverse to develop the concept and meaning of a rational exponent.

Investigate

- Students will likely discover that they can enter the exponent as a decimal instead of a fraction. This works for terminating fractions but can cause problems if approximations are entered for repeating fractions. When this issue arises take a minute or two to investigate. You might investigate a series of calculations as a class, such as:
$9^{0.3} = \ldots$
$9^{0.33} = \ldots$
$9^{0.333} = \ldots$
$9^{0.3333} = \ldots$

Investigate Responses (pages 296-297)

1. a) To multiply a value times itself is to square the value. The inverse of this operation is taking a square root. If a number times itself has a value of 4 then its value must be the square root of 4. I predict that $4^{\frac{1}{2}} = 2$. This is verified when checked with a calculator.

b) By this argument I predict that $9^{\frac{1}{2}} = 3$, $16^{\frac{1}{2}} = 4$, and $25^{\frac{1}{2}} = 5$. These values may be confirmed with a calculator.

2. a) Raising a base to the exponent $\frac{1}{2}$ is equivalent to finding the square root of the base.

b) The need for parentheses depends on the model of calculator and the form chosen to input the fraction. On some calculators it may be possible to input the value in fraction form using the fraction keys without the use of parentheses. If the fraction is input as a division statement the chance of a correct answer becomes much smaller because the calculator is likely to accept only the numerator as the exponent and then divide the resulting power by the denominator of the exponent. As a general rule using parentheses after the power key will provide the correct value.

3. I predict that $36^{\frac{1}{2}} = 6$ and $49^{\frac{1}{2}} = 7$. These values are confirmed with a calculator.

4. a) I can express

$$8^1 = 8^{\frac{1}{3}} + \frac{1}{3} + \frac{1}{3}$$
$$= 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}}$$

In this case I require three identical values with a product of 8. It follows that $8^{\frac{1}{3}} = 2$. The expression $8^{\frac{1}{3}}$ is equivalent to the cube root of 8.

b) I predict that $27^{\frac{1}{3}} = 3$ and $64^{\frac{1}{3}} = 4$. These values are confirmed with a calculator.

5. $\sqrt[4]{16} = 16^{\frac{1}{4}}$

I predict a value of 2 for these expressions, which is confirmed with a calculator.

6. $\sqrt[n]{m} = m^{\frac{1}{n}}$. The *n*th root of a base *m* may be expressed as *m* raised to the reciprocal of the value *n*. This statement is true for any natural number, *n*. Negative or rational values are not defined by this argument.

7. a) $\sqrt[12]{2} = 2^{\frac{1}{12}}$
$= 1.059\ 463$

b) If I raise the answer I obtained in part a) to the power 12 I get the value 2. This is the result I expect. If $2^{\frac{1}{12}} = n$, then
$n^{12} = \left(2^{\frac{1}{12}}\right)^{12}$
$= 2$

Examples

- The key concept behind an expression such as $27^{\frac{2}{3}}$ is that it represents two operations: the extraction of a root and the evaluation of a power. The order the operations are performed in does not matter; the answer is the same. It is usually easiest to determine the root first.
- When using a calculator to evaluate such an expression, it is necessary to enter brackets around the rational exponent. Consider demonstrating, or having students demonstrate, the difference in the answer when brackets are not included. Explain the difference by referring to the order of operations.
- When working through **Example 1**, take some time to reflect on when certain roots do not exist, with reference to a negative base.
- Some calculators will return an error for any negative bases raised to rational exponents, even those that exist. They evaluate the expression by first taking the logarithm of the base, which does not exist for negative bases. In this case, students will need to predict the sign of the answer, and then use their calculator to evaluate a similar question with a positive base. Higher-end scientific calculators and graphing calculators do not have this problem.
- **Example 2** mixes negative bases and negative exponents. Watch for errors in applying the exponent laws. When checking answers with a graphing calculator, it is particularly important to make correct use of the negative key as opposed to the subtraction key. For example, in the expression $\left(-\frac{9}{25}\right)^{\frac{-3}{2}}$, both negative signs are entered using the negative key. The exponent must be entered inside brackets.
- **Example 3** introduces a practical example of the use of rational exponents. This method of tuning a piano makes chords and harmony possible. The same tuning is used on most stringed instruments, such as a guitar or a violin. Although other tunings are possible, they do not result in the rich harmony that characterizes music composed for this type of tuning. If you have experience with music and have access to a keyboard, you could demonstrate octaves, consonant intervals (those that sound pleasant to our ears), and dissonant intervals.
- Whenever possible, ask students to estimate their answers before calculating.

Communicate Your Understanding

- There is more than one way to answer **question C1**. Have students compare different methods and explain their reasoning.
- For the expression $(-8)^{\frac{1}{3}}$ in **question C2**, some calculators will return the correct answer if the exponent is entered as a fraction, but not if the exponent is entered as an approximation of the equivalent decimal. Ask students with different models to try this, and compare the answers.
- Note that the expression $0^{\frac{1}{2}}$ in **question C3** does not return an error. When discussing why this makes sense, take time to revisit the meaning of

rational exponents, and how they arise from the exponent rules.

- You may wish to use **BLM 6–6 Section 6.3 Investigate Rational Exponents** for remediation or extra practice.

Communicate Your Understanding Responses (page 302)

C1 To predict the value of $9^{-\frac{1}{2}}$, factor a negative 1 from the exponent and apply the power of a power property.

$$\begin{aligned} 9^{-\frac{1}{2}} &= \left(9^{\frac{1}{2}}\right)^{-1} \\ &= 3^{-1} \\ &= \frac{1}{3} \end{aligned}$$

A calculator provides the decimal equivalent of this value. In general when I have a negative fraction as an exponent I expect to get the reciprocal of the root I would have obtained with the same fraction in its positive form. For the example above I know that $9^{\frac{1}{2}} = 3$. By making the exponent negative, I got the reciprocal of this value.

C2 According to my calculator, $(-8)^{\frac{1}{3}} = -2$ but $(-4)^{\frac{1}{2}}$ is not defined. In the first case, I can reverse the procedure, $(-2)^3 = -8$. In the second case I require a value for which $x^2 = -4$. This is not possible because any real number raised to the exponent 2 must have a positive value. I get an error message from the calculator as a result of this.

C3 By the definition of a fractional exponent, the value of $0^{\frac{1}{2}}$ must be a number that gives a product of 0 when multiplied by itself, that is, the square root of 0. I know that 0 times itself is 0 and have previously accepted 0 as a value for $\sqrt{0}$. The calculator confirms my prediction, giving a value of 0 rather than an error message.

Practise, Connect and Apply, Extend

- For **questions 1 to 5**, ask students to predict the sign of the answer before evaluating, for problems that involve negative bases and/or exponents.
- For **questions 1 and 2**, consider asking students to check each answer using a calculator.
- For **questions 3 and 4**, ask students to identify the two operations that take place in each part of the question.
- Some calculators will give an error message for problems such as **question 6**, part c). Students will need to predict the sign of the answer and then use their calculator to evaluate a similar question with a positive base.
- The square-cube law in **question 9** turns up in many different contexts. Small animals are much stronger for their size than large animals. Cats can fall a distance many times their own length without breaking any bones, whereas humans risks serious injury from a fall even twice their own height.
- Many students find the answer to **question 10** almost counter-intuitive. You can demonstrate that it is correct using two containers that have the same shape and volumes in the ratio 2:1, such as milk cartons.
- For **question 12**, Kleiber's Law is not very well known. You can find more information, including graphs, by going to *www.mcgrawhill.ca/functionsapplications11* and following the links.

Literacy Connections

- *The square-cube law*, *gravitational acceleration*, and *metabolic rate* are terms that students may not be familiar with. Have students use context clues from each problem and what they may already know about the words that make up the terms to infer a meaning for each term.

Common Errors

- Some students may forget to include the brackets when entering rational exponents on a calculator.

R_x Have students estimate the expected answer whenever possible. For example, $30^{\frac{1}{3}}$ should be slightly greater than 3. Leaving out the brackets returns an answer of 10.

- Some students will have difficulty determining the correct sign for the answer, especially if their calculators cannot handle negative bases.

R_x Ask students to predict the sign of the answer whenever a question involves negative bases or exponents.

- Some students multiply the base by the exponent. For example:

$$16^{\frac{3}{4}} = \overset{4}{\cancel{16}} \times \frac{3}{\underset{1}{\cancel{4}}}$$
$$= 12$$

R_x Present this solution and ask students to explain why it is not correct and how it can be corrected.

Accommodations

Gifted and Enrichment–students could set up a pendulum to test the formula in **question 13**. Ask them why their results might not match the formula exactly.

Memory–ask students to write a note to remind themselves to check the signs of all their answers to **questions 1** through **5**. Is each answer positive or negative, as required?

ESL–many of the problems in this section have elaborate and interesting contexts. Work with students, or have them work with a classmate, to summarize the context of each problem and identify what is needed to solve it.

Student Success

- Estimating each answer before calculating will help students confirm they have used integer exponents and the order of operations correctly.

Mathematical Processes Integration

The table shows questions that provide good opportunities for students to use the mathematical processes.

Process Expectations	Selected Questions
Problem Solving	9–14
Reasoning and Proving	9, 10, 12, 14, 16
Reflecting	12, 13
Selecting Tools and Computational Strategies	1–6, 9–17
Connecting	9–11
Representing	7, 14
Communicating	16