Chapter 4 Solve Exponential Equations

4.1 The Exponent Laws

KEY CONCEPTS

- An algebraic expression in the form a^m is called a power. It consists of the base, a, and the exponent, m. It is a product of identical factors, where the base is the identical factor and the exponent is the number of times the factor appears. For example,

  $$2^3 = 2 \times 2 \times 2$$

  exponential form (2^3) ↗ expanded form ($2 \times 2 \times 2$) ↖

- The exponent of a power may be an integer or a rational number.
- The exponent laws can be used to simplify expressions involving powers with the same base.
  - When multiplying powers of the same base, add the exponents and the base remains the same:
    $a^m \times a^n = a^{m+n}$
  - When dividing powers of the same base, subtract the exponents and the base remains the same:
    $a^m \div a^n = a^{m-n}$
  - When determining a power of a power, multiply the exponents and the base remains the same:
    $(a^m)^n = a^{m \times n}$
  - For a power of a product or quotient, the exponent can be applied to each factor inside the bracket:
    $(ab)^m = a^m b^m$ and $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
- The algebraic expression $a^1 = a$.
- The algebraic expression $a^0 = 1$ for $a \neq 0$.
- A negative exponent results in the reciprocal of the base raised to the corresponding positive exponent: $a^{-n} = \frac{1}{a^n}$ for $a \neq 0$.
- The algebraic expression $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$.

Example

Consider the expression $\frac{a^5b^3c^4}{\sqrt{a^4b^2}}$.

a) Evaluate the expression by substituting $a = 3$, $b = 4$, and $c = -2$ into the expression.

b) Use the exponent laws to simplify the expression, and then evaluate the simplified expression for $a = 3$, $b = 4$, and $c = -2$.

Solution

a)
$$\begin{aligned}\frac{a^5b^3c^4}{\sqrt{a^4b^2}} &= \frac{(3)^5(4)^3(-2)^4}{\sqrt{(3)^4\,(4)^2}}\\ &= \frac{243(64)(16)}{\sqrt{81(16)}}\\ &= \frac{248\,832}{\sqrt{1296}}\\ &= \frac{248\,832}{36}\\ &= 6912\end{aligned}$$

b)
$$\begin{aligned}\frac{a^5b^3c^4}{\sqrt{a^4b^2}} &= \frac{a^5b^3c^4}{(a^4b^2)^{\frac{1}{2}}}\\ &= \frac{a^5b^3c^4}{a^{4\times\frac{1}{2}}\,b^{2\times\frac{1}{2}}}\\ &= \frac{a^5b^3c^4}{a^2b^1}\\ &= a^{5-2}b^{3-1}c^4\\ &= a^3b^2c^4\\ &= a^3b^2c^4\end{aligned}$$

Substitute 3 for a, 4 for b, and -2 for c.

$$\begin{aligned}a^3b^2c^4 &= (3)^3(4)^2(-2)^4\\ &= 27 \times 16 \times 16\\ &= 6912\end{aligned}$$

A

1. Use the exponent laws to simplify. Leave your answers in exponential form.

   a) $(3^3)^6$

   b) $6^{12} \div 6^9$

   c) $\frac{(0.25)^7}{(0.25)^3 \times (0.25)^2}$

   d) $(x^3)(x^4)$

   e) $(-3ab^2)^3$

   f) $\left(\frac{2u^3v^2}{5uv}\right)^2$

2. Simplify. Express your answers using only positive exponents.

   a) $a^{-5} \times a^2$

   b) $(7m^{-4})(-2m^{-3})$

   c) $v^{-3} \div v^{-8}$

   d) $\frac{-10q^{-7}}{-2q^{-3}}$

   e) $(h^{-3})^2$

   f) $(2cd^{-3})^{-4}$

   g) $\left(\frac{3}{mn}\right)^{-2}$

   h) $\left(\frac{5a^2}{3b^4}\right)^{-3}$

3. **a)** Given that $\left(\frac{2}{3}\right)^{-1} = \frac{1}{\frac{2}{3}}$, show that this can also be rewritten as $\frac{3}{2}$.

   b) Use your process from part a) to express $\left(\frac{4}{5}\right)^{-3}$ as a fraction.

   c) Use parts a) and b) to make a general statement about writing $\left(\frac{a}{b}\right)^{-n}$ without negative exponents.

4. Evaluate using a calculator. Round your answers to four decimal places, when necessary.

   a) 4^{-2}

   b) -3.2^{-2}

   c) 1.02×3.4^4

   d) $\sqrt[4]{2.5^3}$

   e) -0.5×2.7^3

   f) $\sqrt[5]{19.7^4}$

5. Simplify. Express your answers using only positive exponents.

   a) $x^{\frac{1}{3}} \times x^{\frac{1}{3}}$

   b) $m^{\frac{1}{4}} \times m^{\frac{2}{3}}$

   c) $s^{\frac{3}{5}} \div s^{-\frac{3}{4}}$

   d) $\dfrac{u^5v}{u^{\frac{3}{7}}v^{-\frac{1}{2}}}$

   e) $\left(y^{\frac{1}{5}}\right)^{-\frac{4}{5}}$

   f) $\left(4c^{\frac{7}{9}}\right)^{-2}$

6. Explain how to evaluate with and without a calculator.

   a) $-\left(\frac{1}{16}\right)^{-\frac{1}{4}}$

   b) $\left(\frac{25}{49}\right)^{-\frac{1}{2}}$

B

7. Bob says that $27^{\frac{2}{3}}$ must be evaluated by taking the cube root of 27, and then squaring the result. Sally says you can first square 27 and then take the cube root. Are they both correct? Does it matter which operation is done first? Explain.

8. **a)** Explain how $\frac{x^3}{x^7}$ can be used to show the meaning of x^{-4}.

   b) Explain how $\frac{a^5}{a^5}$ can be used to show why $a^0 = 1$.

9. **a)** For the following statements, state whether they are true for all possible numbers. Consider positive and negative integers as well as rational numbers.

   i) $\sqrt{x^4} = x^2$ **ii)** $\sqrt{x^2} = x$

   iii) $\sqrt[3]{x^6} = x^2$ **iv)** $\sqrt[3]{x^9} = x^3$

   b) Which of the above is *not* true for all values of the variable? Why is this the case? What change can be made to the statement to make it true?

10. **a)** Express 64 as a power with a base of 2.

    b) Express 64 as a power with a base of 4.

    c) Express 64 as a power with a base of 8.

    d) Express 64 as a power with a base of 16.

11. For each of the following, determine the value(s) of the variable.

    a) $x^5 = 100$

    b) $a^6 = 30$

    c) $b^{1.5} = 20$

12. Use the exponent laws to show that $9^3 = 27^2$.

☆ **13.** Consider the table shown.

n	2^n	Result
3	2^3	8
2	2^2	4
1	2^1	2
0		
–1		
–2		
–3		

a) In row 3, what operation is done as you move from one term to the next?

b) Copy and complete the table.

c) Use the table to explain why $2^0 = 1$ and why $2^{-2} = \frac{1}{2^2}$.

☆ **14. a) Use Technology** Copy the table and then complete it using a scientific calculator.

n	$\sqrt{n}$	$n^{\frac{1}{2}}$
1		
4		
9		
16		

b) What do you notice about the results for $\sqrt{n}$ and $n^{\frac{1}{2}}$, and what conclusion can you make about the relationship between $\sqrt{n}$ and $n^{\frac{1}{2}}$?

c) Does this relationship hold true for all values of n?

d) Use Technology Copy the table and then complete it using a scientific calculator.

n	$\sqrt[3]{n}$	$n^{\frac{1}{3}}$
1		
8		
27		
64		

e) What do you notice about the results for $\sqrt[3]{n}$ and $n^{\frac{1}{3}}$, and what conclusion can you make about the relationship between $\sqrt[3]{n}$ and $n^{\frac{1}{3}}$?

f) Does this relationship hold true for all values of n?

g) Does this relationship hold true for all exponent values?

C

15. a) What is the formula for the volume of a cone?

b) Rewrite this formula to express the radius as a function of the volume and the height of the cone.

c) Use the formula from part b) to determine the radius of a cone with a height of 5 cm and a volume of 35 cm^3. Round your answer to the nearest tenth of a centimetre.

16. When a certain gas expands, the relationship between the pressure, P, in kilopascals (kPa) and the volume, V, in cubic metres is given by $P^3V^2 = 750$.

a) Solve the equation for P.

b) Use the formula from part a) to determine the pressure of the gas when the volume of the gas is 8 m^3.

c) Solve the equation for V.

d) Use the formula from part c) to determine the volume of gas when the pressure of the gas is 5 kPa.