KEY CONCEPTS

- Factoring a polynomial is the opposite of expanding a polynomial.
- The greatest common factor (GCF) is the greatest number and/or variable that is a factor of each term in a polynomial expression.
- To factor a polynomial expression using the method of common factoring, remove the GCF as the first factor, and then divide each term by the GCF to obtain the second factor.
  $8x^2y^3 - 12x^4y = 4x^2y(2y^2 - 3x^2)$
- To factor a polynomial expression using the method of factoring by grouping, factor groups of two terms with a common factor to produce a binomial common factor.
  $$\begin{aligned} bx + 3x + by + 3y &= (bx + 3x) + (by + 3y) \\ &= x(b + 3) + y(b + 3) \\ &= (b + 3)(x + y) \end{aligned}$$
- Quadratic polynomial expressions of the form $ax^2 + bx + c$ can sometimes be factored by applying the method of decomposition. To find the terms to use when decomposing the linear term of an expression of the form $ax^2 + bx + c$, look for the pair of integers whose sum is b and whose product is ac. When $a = 1$, the middle steps of the method of decomposition can be omitted.
  For $6x^2 - 5x + 1$, $a = 6$, $b = -5$, and $c = 1$. Two integers whose product is 6 and whose sum is −5 are −2 and −3.
  $$\begin{aligned} 6x^2 - 5x + 1 &= 6x^2 - 2x - 3x + 1 \\ &= (6x^2 - 2x) + (-3x + 1) \\ &= 2x(3x - 1) - 1(3x - 1) \\ &= (3x - 1)(2x - 1) \end{aligned}$$
- When factoring a polynomial expression, there may be more than one type of factoring. If there is a greatest common factor for the terms in the polynomial expression, apply the method of common factoring, and then apply any other types of factoring as applicable to factor the polynomial expression completely.

Example

Factor completely.

a) $24x^3 + 30x^2 - 6x$

b) $mx + my + nx + ny$

c) $a^2 + 3a + 2$

d) $b^2 + 2b + 1$

e) $3d^2 + 10d + 3$

f) $4x^2 + 8x - 12$

g) $2x^3 + 5x^2 - 12x$

Solution

a) The GCF of the coefficients, 24, 30, and −6, is 6. The GCF of the variable parts, x^3, x^2, and x, is x. Therefore, the GCF of the polynomial is $6x$.

$24x^3 + 30x^2 - 6x = 6x(4x^2 + 5x - 1)$

b) $mx + my + nx + ny = m(x + y) + n(x + y)$ — Note m is common to the first two terms,

$= (x + y)(m + n)$ — while n is common to last two terms. $(x + y)$ is common to both expressions.

c) Find two integers whose product is 2 and whose sum is 3. The integers are 1 and 2.

$a^2 + 3a + 2 = a^2 + a + 2a + 2$ — Break up $3a$ into $a + 2a$.

$= a(a + 1) + 2(a + 1)$ — Factor by grouping.

$= (a + 1)(a + 2)$

d) Find two integers whose product is 1 and whose sum is 2. The integers are 1 and 1.

$b^2 + 2b + 1 = (b + 1)(b + 1)$

$= (b + 1)^2$

e) Find two integers whose product is 9 and whose sum is 10. The integers are 1 and 9.

$3d^2 + 10d + 3 = 3d^2 + d + 9d + 3$ — Break up $10d$ into $d + 9d$.

$= 3(3d + 1) + 3(3d + 1)$ — Factor by grouping.

$= (d + 1)(3d + 1)$

f) First, remove the GCF, and then proceed as before.

$4x^2 + 8x - 12 = 4(x^2 + 2x - 3)$

$= 4(x - 1)(x + 3)$

g) First, remove the GCF, and then proceed as before.

$2x^3 + 5x^2 - 12x = x(2x^2 + 5x - 12)$

$= x(2x^2 + 8x - 3x - 12)$

$= x[(2x^2 + 8x) + (-3x - 12)]$

$= x[2x(x + 4) - 3(x + 4)]$

$= x(x + 4)(2x - 3)$

A

1. Factor each polynomial expression by finding the greatest common factor.
   a) $14x^2 + 21x - 7$
   b) $5a^5 - 16a^3$
   c) $3b^3 - 9b$
   d) $5h^3 - 15h^2$
   e) $8u^5v^2 + 4u^3v^4$
   f) $9m^4 - 6m^3 + 12m^2$

2. Factor each polynomial expression completely.
   a) $ax + ay + bx + by$
   b) $cx + cy - dx - dy$
   c) $x^2 + xy + xy + y^2$
   d) $a^2 - ab - ab - b^2$

3. Factor each polynomial expression completely.
   a) $x^2 + 7x + 10$
   b) $j^2 + 12j + 27$
   c) $k^2 + 5k + 4$
   d) $m^2 - 7m + 10$
   e) $y^2 - 5y + 4$
   f) $r^2 - 8r + 12$

B

4. Factor each polynomial expression completely. Verify using a computer algebra system, if it is available.
   a) $x^2 + 2x - 3$
   b) $a^2 - 5a + 6$
   c) $m^2 + 10m + 16$
   d) $d^2 + 5d - 24$
   e) $w^2 - 7w - 30$
   f) $b^2 + 8b + 15$

5. a) Factor each polynomial expression completely.
   i) $m^2 + 8m + 16$
   ii) $q^2 - 10q + 25$
   iii) $d^2 + 20d + 100$
   iv) $v^2 - 4v + 4$
   v) $s^2 - 12s + 36$
   vi) $r^2 + 6r + 9$

   b) These are all perfect square trinomials. Identify and describe the pattern for recognizing perfect square trinomials.

☆6. Factor each polynomial expression completely.
   a) $3x^2 - 3x - 18$
   b) $4a^2 - 20a - 56$
   c) $p^3 + 8p^2 + 15p$
   d) $dm^2 - 9dm + 14d$
   e) $3ht^2 + 12ht + 9h$
   f) $m^3t^2 - 9m^3t + 20m^3$

7. For question 6a), Sara did not find a common factor and came up with the result $(3x + 6)(x - 3)$. She verified her answer by expanding. Is her factored form correct? Explain your reasoning.

8. Factor each polynomial expression completely.
   a) $3x^2 + 15x - 18$
   b) $4a^3 - 16a^2 + 12a$
   c) $5b^3 - 15b^2 + 20b$
   d) $y^3 - y^2 - 2y$
   e) $10m^4 + 4m^3 - 8m^2$
   f) $6z^5 - 12z^3 + 18z$

☆**9.** Factor each polynomial expression completely.

a) $2m^2 + 11m + 12$

b) $12a^2 - a - 6$

c) $6m^2 + 7m - 3$

d) $4m^2 + 5m - 6$

e) $20z^2 - 23z + 6$

f) $8b^2 - 26b + 15$

10. The area of a rectangular deck is modelled by the polynomial expression $15x^2 + 26x + 8$.

a) Determine expressions for the length and width of the rectangular deck.

b) For what value of x will the deck be a square?

11. The volume of a box is modelled by the polynomial expression $x^3 + 3x^2 + 2x$, with the height of the box being the shortest dimension. Determine expressions for the length, the width, and the height of the box.

12. Beach volleyball is played by two teams of two players on a sand court with area given by $3x^2 + 10x + 3$.

a) Determine expressions for the length and width of the sand court.

b) Determine the actual dimensions of the sand court if x is 5 m.

13. A regulation basketball court in international games has an area given by $3x^2 + 25x + 28$.

a) Determine expressions for the length and width of the basketball court.

b) Determine the actual dimensions of the basketball court if x is 8 m.

14. Canadian football is played on a field that has an area given by $21x^2 + 65x + 50$.

a) Determine expressions for the length and width of the football field.

b) Determine the actual dimensions of the football field if x is 20 yd.

15. The area of a swimming pool used for the Olympic Games is given by $21x^2 + 31x + 4$.

a) Determine expressions for the length and width of the swimming pool.

b) Determine the actual dimensions of the swimming pool if x is 7 m.

c) If there are 9 lane ropes evenly spaced running the length of the pool, determine the width of a lane.

16. A volleyball is hit into the air. Its path can be approximated by the formula $h = -5t^2 + 23t + 10$, where h is the height, in centimetres, of the volleyball above the ground and t is the time, in seconds, since it was hit.

a) Write the formula in factored form.

b) What is the height of the volleyball after 4 s?

17. The total revenue from sales of fleece jackets can be modelled by the expression $720 + 4x - 2x^2$, where x represents the number of jackets sold above the minimum needed to break even. Revenue is also calculated as the product of the number of jackets sold and the price per jacket.

a) Determine the number sold.

b) Determine the price per jacket.

Hint: As the price increases, the number sold decreases.

C

18. Factor completely.

a) $x^3 + 2x^2 - x - 2$

b) $x^3 + 3x^2 - 4x - 12$

19. Factor completely.

a) $x^4 - 5x^2 + 4$

b) $(x^2 - x)^2 - 8(x^2 - x) + 12$