8.7 Volume of a Sphere

Strand:
Measurement and Geometry

Strand:
Number Sense and Algebra

Student Text Pages
462–469

Suggested Timing
80 min

Tools
- cylindrical containers that each just hold three tennis balls
- three tennis balls for each group of students
- water
- containers to catch the overflow water
- identical small spheres (e.g., marbles, tennis balls, or table tennis balls)

Technology Tools
- graphing calculators
- *The Geometer's Sketchpad®*
- computers

Related Resources
BLM 8.7.1 Practice: Volume of a Sphere
BLM A6 Knowledge/Understanding General Scoring Rubric
BLM 8.7.2 Achievement Check Rubric
BLM T4 *The Geometer's Sketchpad®* 3
BLM T5 *The Geometer's Sketchpad®* 4

Mathematical Process Expectations Emphasis

- ☑ Problem Solving
- ☑ Reasoning and Proving
- ☑ Reflecting
- ☑ Selecting Tools and Computational Strategies
- ☑ Connecting
- ☑ Representing
- ☑ Communicating

Specific Expectations

Solving Problems Involving Perimeter, Area, Surface Area, and Volume

MG2.04 develop, through investigation (e.g., using concrete materials), the formulas for the volume of a pyramid, a cone, and a sphere (e.g., use three-dimensional figures to show that the volume of a pyramid [or cone] is the volume of a prism [or cylinder] with the same base and height, and therefore that $V_{\text{pyramid}} = \dfrac{V_{\text{prism}}}{3}$ or $V_{\text{pyramid}} = \dfrac{(\text{area of base})(\text{height})}{3}$;

MG2.06 solve problems involving the surface areas and volumes of prisms, pyramids, cylinders, cones, and spheres, including composite figures.

Operating With Exponents

NA1.01 substitute into and evaluate algebraic expressions involving exponents (i.e., evaluate expressions involving natural-number exponents with rational-number bases [e.g., evaluate $\left(\frac{3}{2}\right)^3$ by hand and 9.8[3] by using a calculator]);

Link to Get Ready

The Get Ready segment Calculate Surface Area and Volume provides the needed skills for this section. You may wish to have students complete question 7 before starting this section.

Warm-Up

1. Determine the volume of the following figures:
   a) a cylinder with radius 8 cm and height 15 cm
   b) a cone with radius 8 cm and height 15 cm
   c) a cylinder with diameter 4 m and height 4 m
   d) a cone with diameter 4 m and height 4 m

Warm-Up Answers

1. **a)** 3015.9 cm^3 **b)** 1005.3 cm^3 **c)** 50.3 m^3 **d)** 16.8 m^3

Teaching Suggestions

- Discuss the hot air balloon photo. The discussion should include the empty space in the large balloon when it is filled with the ordinary soccer balls. Ask students, *Is the volume of the hot air balloon the same as the volume of the soccer balls inside it? Why or why not?* (5 min)
- Assign the Investigate. (10–20 min) Since it involves the displacement of water, you will need a large container for the overflow of water. Consider performing this Investigate in a science lab where sinks are available. Have students work in small groups for the Investigate, or demonstrate it for the class. The Investigate should lead to the conclusion that the volume of the sphere is $\frac{2}{3}$ the volume of the cylinder.
- Work through Example 1 with the class to develop the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$.
- Assign Example 2. (10–15 min)

Common Errors

- Some students may have trouble with the formula because of the fraction involved.

R_x Ensure that students realize that $V = \frac{4}{3}\pi r^3$ is the same as $V = \frac{4\pi r^3}{3}$, is the same as $V = \frac{4\pi}{3}r^3$. They may need help using their calculators at first. Using an overhead graphing calculator to demonstrate the different ways the values can be entered may be helpful.

Ongoing Assessment

- Use Achievement Check question 11 to monitor student success. See Achievement Check Answers and **BLM 8.7.2 Achievement Check Rubric**.
- Question 6, the Chapter Problem question, can also be used as an assessment tool.
- Communicate Your Understanding questions can be used as quizzes to assess students' communication skills.

Accommodations

Gifted and Enrichment—Challenge students to investigate the aerodynamics related to hot air balloons.

Motor—Encourage students to work with a partner when working through the Investigate in this section.

Memory—Allow students to use formulas when completing quizzes and tests.

- Assign and discuss the Communicate Your Understanding questions. (10 min)
- Assign Practise questions 1 to 4.
- You may wish to use **BLM 8.7.1 Practice: Volume of a Sphere** for remediation or extra practice.

Investigate Answers (pages 462–463)

1. Answers will vary. Students will likely estimate the diameter to be about 6.5 cm. Not knowing any formula for a sphere, they might consider the volume of a cube with sides 6.5 cm and estimate that the sphere would fill about $\frac{2}{3}$ of that space. So, $\frac{2}{3} \times 6.5^3$ is about 183 cm^3. Or, they might compare the volume of the sphere to the volume of the cylinder and estimate that it would fill a fraction of that space. Anything in the ballpark from 125 to 225 cm^3 would be reasonable. (The actual volume is about 143.79 cm^3.)

2. a) The diameter of the cylinder is about 7 cm.
b) The height of the cylinder is about 21 cm.

5. The height of the displaced water is about 14 cm.

6. $\frac{2}{3}$; Volume of the displaced water is equal to the volume of the three tennis balls.

7. $\frac{2}{3}$

8. Reflect: The volume of a sphere is two-thirds the volume of a cylinder with the same radius and a height equal to the diameter of the sphere.

Volume of the sphere $= \frac{2}{3} \times$ volume of the cylinder

To find the volume of one tennis ball, divide the volume of the displaced water by 3.

9. Answers will vary.

Communicate Your Understanding Responses (page 465)

C1. Step 1: Use the formula for the surface area of a sphere to find the radius.
Step 2: Substitute the value of the radius into the formula for the volume of a sphere.

C2. If you doubled the volume of a sphere, its volume would be eight times the original volume.

Practise

Practise questions 1 to 4 are similar to Examples 1 and 2. Students should not have difficulty if the Examples are covered with the class before assigning these questions.

Connect and Apply

Question 5 requires that students use proportional reasoning to calculate the mass of the lollipop. This assumes that both lollipops have the same density and that the largest one was also spherical.

Question 6, the Chapter Problem question, is a sphere just fitting inside a cylinder. The students must assume that the dimensions given are of the interior of the cylinder. You might discuss with the class that this cylinder might be made of packing foam to protect the glass gazing ball that it holds. Most students will have experienced packages like this.

Question 7 is similar, but here the spheres are in a rectangular prism package. Having a package of golf balls in the classroom may help students visualize the problem. This question does not require a volume calculation, but an extension would be to calculate the empty space in the box by calculating the volume of the golf balls and the volume of the box.

Question 8 involves a silo. This question also requires use of percent skills and calculating the number of truckloads required to fill the silo to 80% capacity. Students should be reminded to round the number of truckloads up (never down) to the nearest whole number in this kind of question. You may wish to use **BLM A6 Knowledge/Understanding General Scoring Rubric** to assist you in assessing your students.

For question 9, help students realize that the two hemispherical ends of the tank truck go together to make a complete sphere, so the volume can be calculated by finding the sum of the volume of the cylinder and the sphere.

Question 10, a Fermi problem, will lead to some great discussion about basketballs filling the room and how much space would be empty. Students could start by imagining layers of basketballs arranged in the room. Lead the class discussion to realize that more basketballs could fit into the room if the spheres fit more closely together. It would be helpful to have several spheres on hand (marbles, tennis balls or table tennis balls) to illustrate this packing technique. The idea here is to choose a reasonable estimation technique. A range of answers is appropriate.

In question 11, the Achievement Check, lead students to recognize different factors that they should take into consideration in part c). For example, amount of wasted space, amount of material required to make the package, ease of stacking and shipping, etc. Either answer is appropriate with proper justification. You may wish to use **BLM 8.7.2 Achievement Check Rubric** to assist you in assessing your students.

Achievement Check Answers (page 467)

11. a) The square-based prism has sides 8.5 cm by 8.5 cm by 17 cm.

$$\begin{aligned}\text{Volume of prism} &= 8.5 \times 8.5 \times 17\\ &= 1228.25\text{ cm}^3\end{aligned}$$

The cylinder has a diameter of 8.5 cm and a height of 17 cm.

$$\begin{aligned}\text{Volume of cylinder} &= \pi \times (4.25)^2 \times 17\\ &= 964.7\text{ cm}^3\end{aligned}$$

b) The amount of empty space in each package is the volume of the package minus the volume of the two tennis balls.

$$\begin{aligned}\text{Volume of 2 tennis balls} &= 2 \times 4 \div 3 \times \pi \times (4.25)^3\\ &= 643.1\text{ cm}^3\end{aligned}$$

$$\begin{aligned}\text{Empty space in the square-based prism} &= 1228.25 - 643.1\\ &= 585.15\text{ cm}^3\end{aligned}$$

$$\begin{aligned}\text{Empty space in the cylinder} &= 964.7 - 643.1\\ &= 321.6\text{ cm}^3\end{aligned}$$

There would be almost twice as much empty space in the square-based prism package.

c) Answers may vary. The cylinder will be a better choice if the point is to leave less empty space in the package. The cylinder is also the better choice if it is important to use less packaging material.

$$\begin{aligned}SA_{\text{cylinder}} &= 2 \times \pi \times (4.25)^2 + 2 \times \pi \times 4.25 \times 17\\ &= 567.5\text{ cm}^2\end{aligned}$$

$$\begin{aligned}SA_{\text{Square-based prism}} &= 2 \times (8.5)^2 + 4 \times 8.5 \times 17\\ &= 722.5\text{ cm}^2\end{aligned}$$

The package design may also vary depending on the style of print and colour that is used to make it catch the eye. (e.g., triangular prism, colour panels).

Extend

Question 12 involves rearranging the volume formula. Some of your students will be quite adept at this rearranging by now, but to solve for the radius it is necessary to take the cube root, which has not been done previously.

For question 13, enter $y = \frac{4}{3}\pi x^3$ into the graphing calculator. Have students choose an appropriate window for this function. This question requires the use of the Trace feature on the graphing calculator. Students will find this question easier if question 9 from Section 8.6 Surface Area of a Sphere was assigned previously. Have students graph both the surface area and the volume functions simultaneously to compare them as an extension activity. Have a discussion about their relative rates of change. You may wish to use the terms *quadratic* and *cubic* to describe these functions, but simply treating the graphs as examples of non-linear relations would suffice. You may wish to tell students that they will study functions like these in more detail in future courses.

Question 14 requires significant algebra skills, but is appropriate for your stronger students. As an extension, have students look at the general case when surface area is doubled. If r represents the original radius and if R represents the radius of the larger sphere,

$$\begin{aligned} 2(4\pi r^2) &= 4\pi R^2 \\ 2r^2 &= R^2 \\ \sqrt{2r^2} &= R \\ \sqrt{2}r &= R \end{aligned}$$

Now, if the volume of the large sphere is considered,

$$\begin{aligned} V &= \frac{4}{3}\pi\left(\sqrt{2}r\right)^3 \\ &= \frac{4}{3}\pi\left(2\sqrt{2}r^3\right) \\ &= 2\sqrt{2}\left(\frac{4}{3}\pi r^3\right) \\ &= 2\sqrt{2}\ \text{(original volume)} \end{aligned}$$

Students would use the decimal approximation for $2\sqrt{2}$, but you could show them that this is the exact value. This would only be appropriate for your best students.

Question 15 is suitable for most students, although the estimating in part a) will be more difficult for weaker students. Again, your better students could be encouraged to look at the general case for this question, which appears in question 16.

Question 17 is a *The Geometer's Sketchpad®* activity that students could try at home with the Student Edition of *The Geometer's Sketchpad®* or in class, if time allows and computers are available. You may wish to use **BLM T4 *The Geometer's Sketchpad®* 3** or **BLM T5 *The Geometer's Sketchpad®* 4** to support this activity.

Question 18, a Math Contest question, is a good question to summarize the volume concepts from this chapter. Most students can handle this question by doing the volume calculations. Encourage your better students to try to answer the question without doing the actual calculations and by taking a more algebraic approach.

Question 19, a Math Contest question, is an extension of question 6, the Chapter Problem question, and would be a great extension to the Investigate where a can of tennis balls was used.

Exercise Guide

Category	Question Number
Minimum (essential questions for all students to cover the expectations)	1–4
Typical	1–5, 7–10
Extension	12–19