8.3 Find Angles Using the Cosine Law

Student Text Pages
412–419

Suggested Timing
80 min

Tools
- protractor
- ruler

Technology Tools
- TI-89 calculator
- computer
- *The Geometer's Sketchpad*®

Related Resources
- G–4 Protractor
- T–4 *The Geometer's Sketchpad*® 3
- T–5 *The Geometer's Sketchpad*® 4
- T–7 The Computer Algebra System (CAS) on the TI-89 Calculator
- BLM 8–8 Section 8.3 Practice Master
- BLM 8–9 Section 8.3 Achievement Check Rubric

TI-Navigator™

Go to **www.mcgrawhill.ca/books/principles10** and follow the links to the file for this section.

Teaching Suggestions

- Discuss the photo and problem posed in the introductory paragraphs. This is an example of a situation in which three pieces of information about a triangle are given, but it is not clear whether it can be solved and, if so, which tool can be applied to do so. (5 min)

Investigate

- Have students discuss the **Investigate** in pairs or small groups and try to brainstorm a solution, using the steps as a guide. Prompt groups who are stuck: (10–15 min)
  - Which tools are at your disposal? (sine law, cosine law, Pythagorean theorem, primary trigonometric ratios)
  - Which may or may not apply? (Pythagorean theorem and the primary trigonometric ratios only apply if there is a right angle)
  - Is there a formula that may apply in which all information is known except one piece?
- Students should recognize that they should use the cosine law because it involves three sides (all given in this case) and one angle (unknown). However, they may struggle with the algebraic manipulation required to effectively do so. The worked examples that follow model this process. Encourage students strong in algebra to solve the problem posed in the introduction. Encourage successful groups to share their methods with the class.

Examples

- Discuss the **Examples** with the class. (25–30 min)
- **Example 1** poses another problem in which three sides of a triangle are given, and interior angles are required. In part a), students see how to rearrange the cosine law to solve for a missing angle. This process is illustrated in three ways.
  - Method 1 involves direct substitution into the cosine law, followed by the application of equation solving techniques that students should be familiar with. This method is less abstract than the one that follows.
  - Method 2 shows the same equation solving steps, but performed prior to substitution. This type of algebraic manipulation is more abstract, but exposure to it will serve students well for future studies in university-destination mathematics courses.
  - Method 3 is conceptually the same as Method 2, with the exception that the CAS performs each algebraic step. This approach still requires the CAS user to instruct the technology at each step in the manipulation. The CAS screen shots shown are of a TI-89 calculator. However, the same logic can be used for other CAS hand-held devices or CAS software, such as *Derive*™. The use of CAS may benefit students who can apply the logical reasoning involved in rearranging the cosine law, but tend to stumble when carrying out the algebraic steps. The CAS solution presents two issues that should be addressed:
  - The division bar spans across both sides of the equation. This is simply

a characteristic of the software, and students should not use this form for written solutions.
  – In the next line, the negative sign seems to disappear from the denominator. The program has actually multiplied numerator and denominator by –1 to express the equation in a cleaner form.

If needed, use **T–7 The Computer Algebra System (CAS) on the TI-89 Calculator** to support this activity.

- Part b) of **Example 1** illustrates that after you have applied the cosine law once to find a fourth piece of information about a triangle, simpler tools (in this case, the sine law) can generally be applied to find any additional required information.
- **Example 2** returns attention to the problem posed in the introductory text on page 412. Students once again see how the cosine law can be represented differently to more easily solve for an unknown angle. The other angles are then found by applying the sine law and geometric reasoning.

Communicate Your Understanding

- Discuss the **Communicate Your Understanding** questions. (5 min)
- These questions can be used to help students consolidate their understanding of when to use the sine law and the cosine law, and whether or not they need to rearrange the cosine law.
- Students should be aware of the logic behind the steps in rearranging the cosine law into its alternative form. They should also realize that the cosine law is typically required just once for any given triangle.
- Use **BLM 8–8 Section 8.3 Practice Master** for remediation or extra practice.

Investigate Answers (page 412)

1. Answers will vary.

2. No. Answers will vary.

3. a) Yes. The cosine law can be rearranged to find the measure of one of the angles in △ABC.

b) The choice of angle does not matter. The three cosine law formulas for △ABC can be rearranged to give the following three formulas.

$a^2 = b^2 + c^2 - 2bc(\cos A)$ can be rearranged to give the formula $\cos A = \dfrac{a^2 - b^2 - c^2}{-2bc}$.

$b^2 = a^2 + c^2 - 2ac(\cos B)$ can be rearranged to give the formula $\cos B = \dfrac{b^2 - a^2 - c^2}{-2ac}$.

$c^2 = a^2 + b^2 - 2ab(\cos C)$ can be rearranged to give the formula $\cos C = \dfrac{c^2 - a^2 - b^2}{-2ab}$.

4. Answers will vary.

5. The cosine law can be used to find an angle measure if three side lengths are known in an acute triangle. The cosine law can be solved and rearranged as follows:

$$a^2 = b^2 + c^2 - 2bc(\cos A)$$
$$a^2 - b^2 - c^2 = -2bc(\cos A)$$
$$\frac{a^2 - b^2 - c^2}{-2bc} = \cos A$$

Common Errors

- Calculator outputs may not make sense.

R_x Have students check that calculators are in degree mode and adjust if necessary.

- Some students may substitute improperly into the cosine law.

R_x Have students practise identifying angles and their corresponding sides by drawing an arrow from the angle across the triangle to identify its corresponding side. Encourage students to label angles using capital letters and their corresponding sides using the corresponding lower case letters.

- Students provide incomplete solutions.

R_x Have students read carefully what is asked for in a question. In particular, they should note that to solve a triangle means to find all of its side lengths and all of its interior angle measures. In most cases students need only apply the cosine law once, after which simpler tools can be used (e.g., sine law, geometric properties).

Accommodations

Gifted and Enrichment—Challenge students to create extra questions and the answers to these questions for their classmates.

Visual—Encourage students to use Method 2 of Example 1 to determine the missing angles (i.e., substitute the values for the sides of the triangle into the rearranged formula).

Perceptual—Let students work in study groups to create solutions to these questions.

Memory—Provide students with cue cards, where each step for solving a triangle given its three sides is written on a separate card, and have students arrange the steps in the correct order for the solution.

Communicate Your Understanding Responses (page 417)

C1. You can use the sine law to find an unknown angle if you have two sides and the angle across from one of the two sides. You can use the cosine law to find the first angle if only the three sides are given. Answers will vary.

C2. Isolate the term containing cos A by subtracting b^2 and c^2 from both sides; isolate cos A by dividing both sides by $-2bc$; simplify; substitute in the given side lengths, 4 for a, 5 for b, and 6 for c; simplify; take the inverse cosine of $\frac{45}{60}$; round the answer for the angle to the nearest degree.

C3. a) Answers may vary. For example: Use the cosine law to solve for the largest angle, ∠V. Then use the sine law to solve for the second largest angle. To find the third angle, use the sum of the interior angles in a triangle is 180°.

b) Answers may vary. For example: You can solve for all the angles using the cosine law.

Practise

- Encourage students to start with the cosine law in its usual form and to rearrange it, either before or after substituting known values. This method is preferred over having students memorize the alternative form of the cosine law because it builds on conceptual understanding.
- In **questions 1** through **3**, students should practise identifying the correct version of the standard cosine law formula to use. In **questions 5** and **6**, any version of the cosine law can be used to find any of the three unknown angles first.
- **Question 4** illustrates the value of using simpler tools after applying the cosine law once.
- If needed, use **T–4 *The Geometer's Sketchpad*® 3** or **T–5 *The Geometer's Sketchpad*® 4** to support **question 7**.
- **Question 8** connects acute angle trigonometry to right angle trigonometry and measurement. Once the measures of the interior angles are found, one of them can be used to find the height. Lastly, find the area by applying the formula $A = \frac{1}{2}bh$.
- **Question 9** connects the angle found using the cosine law to an exterior angle of the triangle. Students need to recall the properties of complementary angles.
- In **question 10**, students apply the meaning of "midpoint" to manipulate some of the given information.
- **Question 15** challenges students' ability to reason algebraically and to see that special cases of geometric situations can lead to simplified algebraic relationships. In this case the cosine law reduces to a simpler formula when the triangle is isosceles.
- **Question 16** is another opportunity for students to apply algebraic and geometric reasoning to derive a simplified result. This value will become increasingly useful in grade 11 when students apply the trigonometry of special triangles.

Student Success

Have students make up a problem, and then give it to a classmate to complete.

Achievement Check Sample Solution, question 14, page 419

Provide students with **BLM 8–9 Section 8.3 Achievement Check Rubric** to help them understand what is expected.

14. a) Yes, because you are given three pieces of information about the triangle, including one side length.

b) Since the lengths of two sides and measure of the contained angle are given, apply the cosine law to find the third side of the triangle. Then, use the sine law to find the measure of the one of the missing angles. Finally, use the fact that the sum of the interior angles in a triangle is 180° to find the measure of the remaining angle.

c) Use the cosine law.

$d^2 = u^2 + p^2 - 2up(\cos D)$

$d^2 = 64^2 + 60^2 - 2(64)(60)(\cos 39°)$

$d^2 = 1727.5190 \ldots$

$d = \sqrt{1727.5190\ldots}$

$d \doteq 42$

Ursala's house is the best choice for a meeting since the two shortest distances lead to her house. This assumes that all students walk at the same pace and that it is possible to walk through the park in straight lines from one house to another.

Mathematical Processes Integration

The table shows questions that provide good opportunities for students to use the mathematical processes.

Process Expectations	Selected Questions
Problem Solving	12, 14–16
Reasoning and Proving	4, 8, 9, 12, 14–16
Reflecting	4, 7, 13, 14
Selecting Tools and Computational Strategies	5, 6, 8, 10, 11, 13–15
Connecting	8–11, 13, 14
Representing	3, 6, 7, 12, 13, 15, 16
Communicating	4, 12–14

Ongoing Assessment

- Use Achievement Check question 14 to monitor student success. See Achievement Check Answers and **BLM 8–9 Section 8.3 Achievement Check Rubric**.
- Chapter Problem question 13 can also be used as an assessment tool.
- Communicate Your Understanding questions can be used as quizzes to assess students' communication skills.