Exploring the Pythagorean Relationship

3.2

***MathLinks 8*, pages 88–94**

Suggested Timing
80–100 minutes

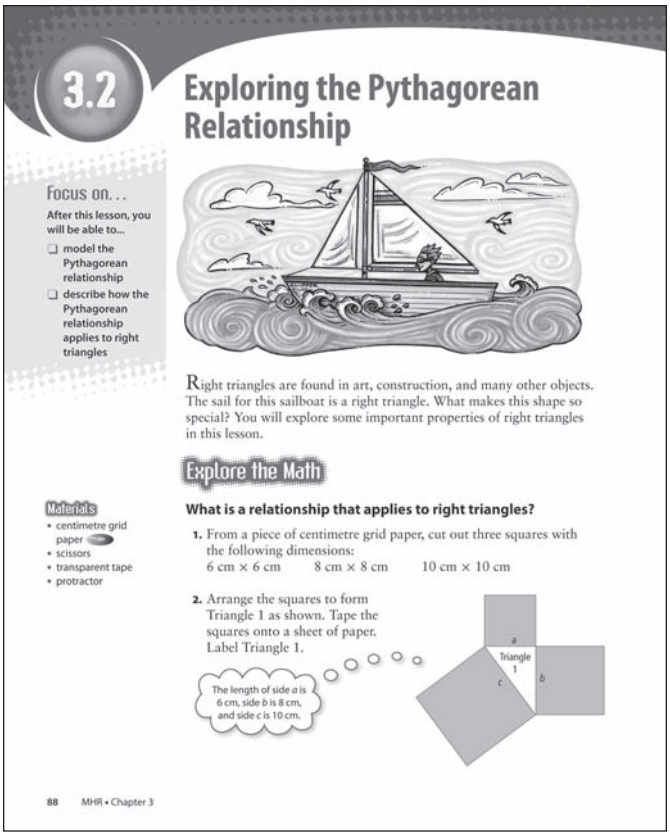
Materials
- centimetre grid paper
- scissors
- transparent tape
- ruler
- protractor

Blackline Masters
Master 5 Tangram
Master 8 Centimetre Grid Paper
BLM 3–3 Chapter 3 Warm-Up
BLM 3–8 Table for Explore the Math
BLM 3–9 Triangles for Explore the Math
BLM 3–10 Section 3.2 Extra Practice
BLM 3–11 Section 3.2 Math Link

Mathematical Processes
- ☑ Communication (C)
- ☑ Connections (CN)
- ☐ Mental Mathematics and Estimation (ME)
- ☑ Problem Solving (PS)
- ☑ Reasoning (R)
- ☑ Technology (T)
- ☑ Visualization (V)

Specific Outcomes
SS1 Develop and apply the Pythagorean theorem to solve problems.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	1–3, 5, 6, 8, 10, 12a), c), Math Link
Typical	1–3, 5, 6, 8, 10, 12–17, Math Link
Extension/Enrichment	1–3, 14, 17, 18–21



Planning Notes

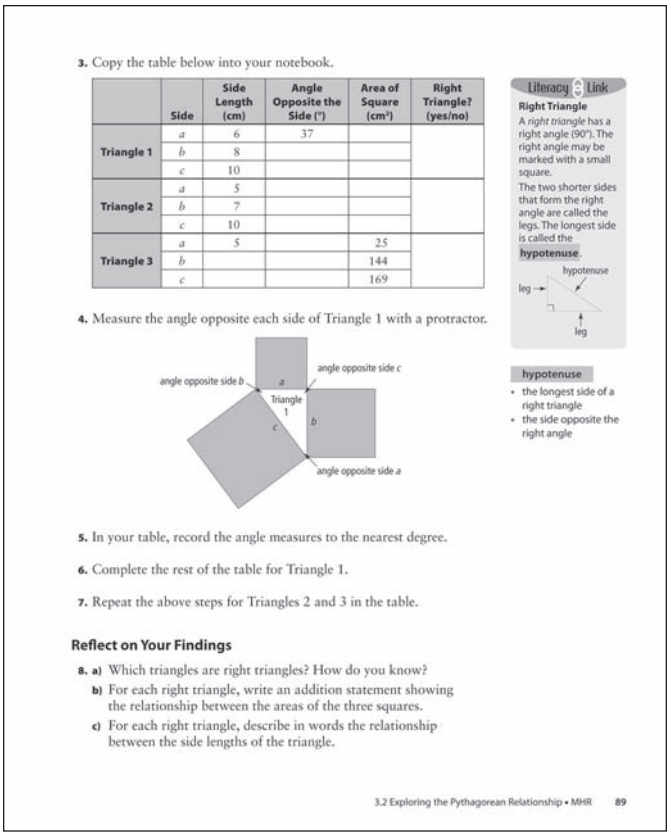
Have students complete the warm-up questions on **BLM 3–3 Chapter 3 Warm-Up** to reinforce material learned in previous sections.

Ask students where right triangles are found in the world, aside from on sailboats. Ask them how they know that these shapes are right triangles.

Explore the Math

This exploration will allow students to create and verify right triangles. The construction of a triangle from three squares helps to reinforce the Pythagorean relationship. Have students copy and complete the table in the student resource, or distribute **BLM 3–8 Table for Explore the Math**, which provides a blank copy of the table on page 89 that can be filled in by students.

Method 1 Have students work in pairs to construct the triangles and make the appropriate measurements from their triangles.



3. Copy the table below into your notebook.

	Side	Side Length (cm)	Angle Opposite the Side (°)	Area of Square (cm²)	Right Triangle? (yes/no)
Triangle 1	*a*	6	37		
	b	8			
	c	10			
Triangle 2	*a*	5			
	b	7			
	c	10			
Triangle 3	*a*	5		25	
	b			144	
	c			169	

Literacy Link

Right Triangle

A *right triangle* has a right angle (90°). The right angle may be marked with a small square.

The two shorter sides that form the right angle are called the legs. The longest side is called the **hypotenuse**.

hypotenuse, leg, leg

hypotenuse

- the longest side of a right triangle
- the side opposite the right angle

4. Measure the angle opposite each side of Triangle 1 with a protractor.

angle opposite side *b*, angle opposite side *c*, Triangle 1, angle opposite side *a*

5. In your table, record the angle measures to the nearest degree.

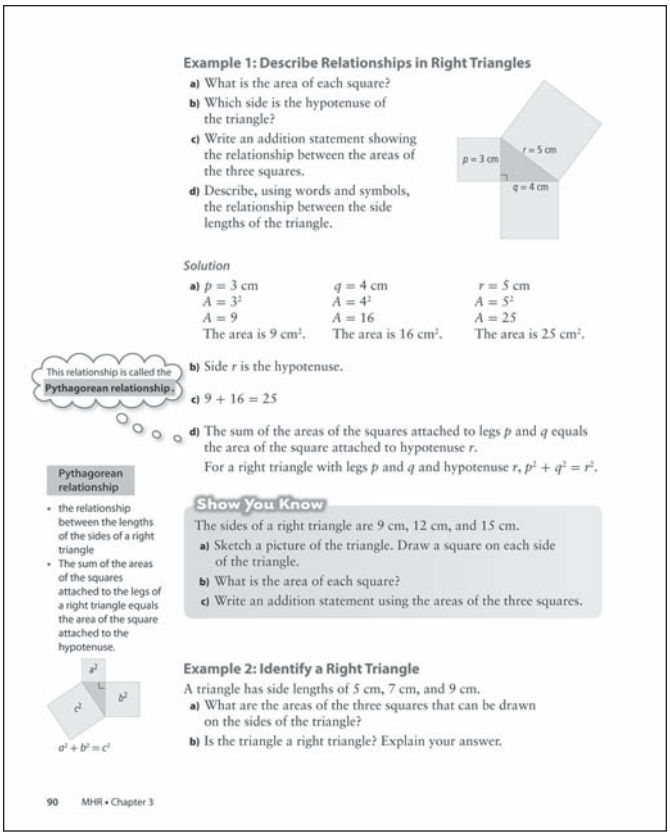
6. Complete the rest of the table for Triangle 1.

7. Repeat the above steps for Triangles 2 and 3 in the table.

Reflect on Your Findings

8. a) Which triangles are right triangles? How do you know?
   b) For each right triangle, write an addition statement showing the relationship between the areas of the three squares.
   c) For each right triangle, describe in words the relationship between the side lengths of the triangle.

3.2 Exploring the Pythagorean Relationship • MHR 89



Example 1: Describe Relationships in Right Triangles

a) What is the area of each square?
b) Which side is the hypotenuse of the triangle?
c) Write an addition statement showing the relationship between the areas of the three squares.
d) Describe, using words and symbols, the relationship between the side lengths of the triangle.

$p = 3$ cm, $r = 5$ cm, $q = 4$ cm

Solution

a) $p = 3$ cm: $A = 3^2$, $A = 9$, The area is 9 cm².
$q = 4$ cm: $A = 4^2$, $A = 16$, The area is 16 cm².
$r = 5$ cm: $A = 5^2$, $A = 25$, The area is 25 cm².

This relationship is called the **Pythagorean relationship**.

b) Side r is the hypotenuse.

c) $9 + 16 = 25$

d) The sum of the areas of the squares attached to legs p and q equals the area of the square attached to hypotenuse r.
For a right triangle with legs p and q and hypotenuse r, $p^2 + q^2 = r^2$.

Pythagorean relationship

- the relationship between the lengths of the sides of a right triangle
- The sum of the areas of the squares attached to the legs of a right triangle equals the area of the square attached to the hypotenuse.

Show You Know

The sides of a right triangle are 9 cm, 12 cm, and 15 cm.
a) Sketch a picture of the triangle. Draw a square on each side of the triangle.
b) What is the area of each square?
c) Write an addition statement using the areas of the three squares.

a^2, b^2, c^2, $a^2 + b^2 = c^2$

Example 2: Identify a Right Triangle

A triangle has side lengths of 5 cm, 7 cm, and 9 cm.
a) What are the areas of the three squares that can be drawn on the sides of the triangle?
b) Is the triangle a right triangle? Explain your answer.

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Method 2 Hand out copies of **BLM 3–9 Triangles for Explore the Math**, which provides diagrams of the three triangles and the attached squares (to scale). Have students make the appropriate measurements from these diagrams.

Literacy Link As a class, discuss the Literacy Link called Right Triangle on page 89. You may wish to have students create a poster called The Pythagorean Relationship for section 3.2. In the centre of the poster, they might draw a diagram of a right triangle and label the hypotenuse and each leg, as in the Literacy Link. As you progress through the section, have students add to the poster. For example, they might add squares off of the sides of the triangle, label the sides with variables, write the Pythagorean relationship underneath the triangle using these variables, etc.

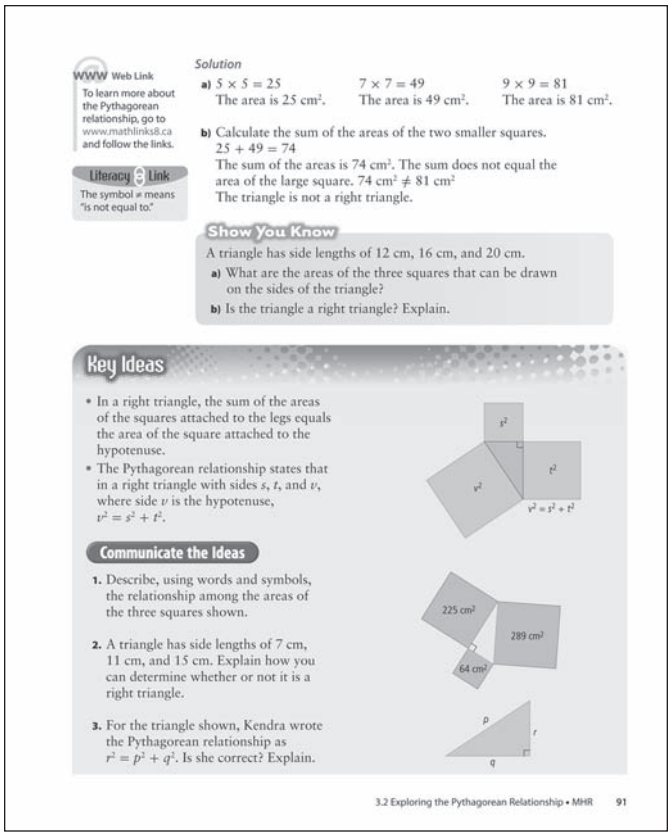
Example 1

The focus of Example 1 is on the relationship of the areas of the squares that form the triangle. The steps in the activity lead students to discover the Pythagorean relationship on their own, rather than being given the equation.

Example 2

In Example 2, students use the Pythagorean relationship to verify that a triangle has a right angle. This activity reinforces the Pythagorean relationship by confirming the connection between the squares of the side lengths of right triangles.

Literacy Link As a class, discuss the Literacy Link on page 91. Have students develop a number statement in which they can use the symbol $\neq$.



WWW Web Link
To learn more about the Pythagorean relationship, go to www.mathlinks8.ca and follow the links.

Literacy Link
The symbol ≠ means "is not equal to."

Solution

a) $5 \times 5 = 25$ The area is 25 cm². $7 \times 7 = 49$ The area is 49 cm². $9 \times 9 = 81$ The area is 81 cm².

b) Calculate the sum of the areas of the two smaller squares.
$25 + 49 = 74$
The sum of the areas is 74 cm². The sum does not equal the area of the large square. 74 cm² ≠ 81 cm²
The triangle is not a right triangle.

Show You Know

A triangle has side lengths of 12 cm, 16 cm, and 20 cm.
a) What are the areas of the three squares that can be drawn on the sides of the triangle?
b) Is the triangle a right triangle? Explain.

Key Ideas

- In a right triangle, the sum of the areas of the squares attached to the legs equals the area of the square attached to the hypotenuse.
- The Pythagorean relationship states that in a right triangle with sides s, t, and v, where side v is the hypotenuse, $v^2 = s^2 + t^2$.

Communicate the Ideas

1. Describe, using words and symbols, the relationship among the areas of the three squares shown.
2. A triangle has side lengths of 7 cm, 11 cm, and 15 cm. Explain how you can determine whether or not it is a right triangle.
3. For the triangle shown, Kendra wrote the Pythagorean relationship as $r^2 = p^2 + q^2$. Is she correct? Explain.

3.2 Exploring the Pythagorean Relationship • MHR 91

Meeting Student Needs

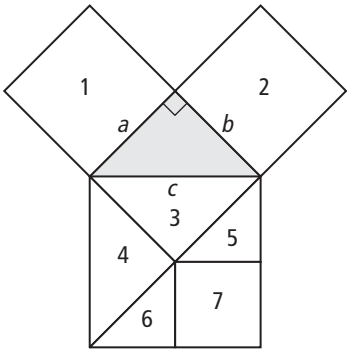
- You may wish to assist some students in recalling how to use a protractor to measure angles.
- It may be beneficial for students to complete at least two additional problems of the type shown in Examples 1 and 2, working first as a whole class and then with a classmate, before doing the Show You Know activity on their own.

ELL

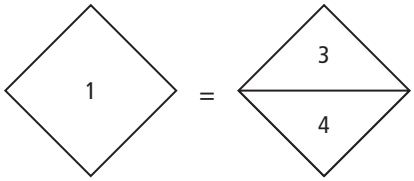
- Ensure that students understand the following terms: *right triangle*, *interior angles*, *addition statement*, *opposite the right angle*, and *longest*.
- You may wish to allow students to answer the Reflect on Your Findings questions in their own language, and then translate their answers into English. Also, consider allowing them to respond visually.
- On the board, model an answer to Explore the Math #8c), which requires students to describe in words the relationship between the side lengths of a right triangle.

Gifted and Enrichment

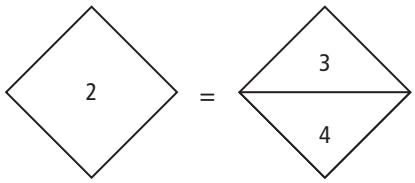
- Have students use tangram pieces to explore a proof of the Pythagorean relationship in an isosceles right triangle (see below). You may wish to provide them with **Master 5 Tangram**.



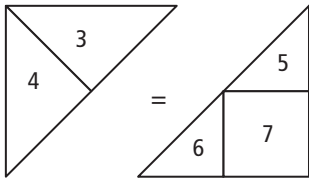
area 1 = a^2
area 2 = b^2
area 3 + 4 + 5 + 6 + 7 = c^2



area 1 = area 3 + 4



area 2 = area 3 + 4



area 3 + 4 = area 5 + 6 + 7

Therefore, area 2 = area 5 + 6 + 7
area 1 + area 2 = area 3
Therefore, $a^2 + b^2 = c^2$

WWW Web Link

You may wish to have students explore the Pythagorean relationship online. Go to www.mathlinks8.ca and follow the links.

Answers

Explore the Math

3.–7.

	Side	Side Length (cm)	Angle Opposite the Side (°)	Area of Square (cm^2)	Right Triangle? (yes/no)
Triangle 1	*a*	6	37	36	yes
	b	8	53	64	
	c	10	90	100	
Triangle 2	*a*	5	28	25	no
	b	7	40	49	
	c	10	112	100	
Triangle 3	*a*	5	23	25	yes
	b	12	67	144	
	c	13	90	169	

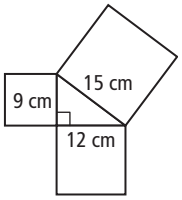
8. a) Triangles 1 and 3 are right triangles. The angle opposite side *c* is 90°.

b) Triangle 1: $36 + 64 = 100$; Triangle 3: $25 + 144 = 169$

c) Answers may vary. Example: In each right triangle, the sum of the areas of the squares attached to the two shorter sides is equal to the area of the square attached to the longest side.

Show You Know: Example 1

a)



b) 81 cm^2, 144 cm^2, 225 cm^2

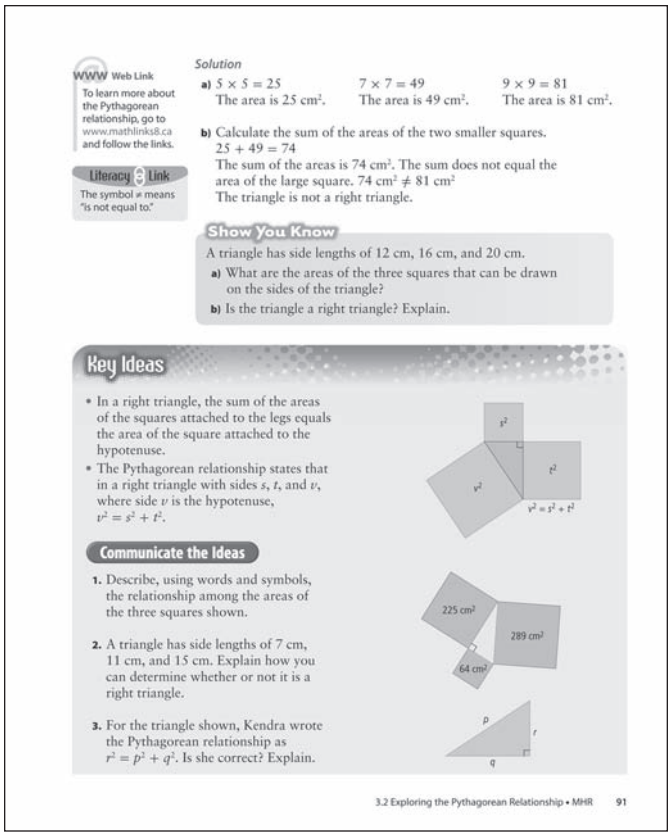
c) $81 + 144 = 225$

Show You Know: Example 2

a) 144 cm^2, 256 cm^2, 400 cm^2

b) Example: Yes, it is a right triangle. The sum of the areas of the smaller squares is equal to the area of the large square: $144 + 256 = 400$.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect on Your Findings Listen as students discuss what they discovered during the Explore the Math. Try to have students generalize the conclusion about their findings.	• Ensure that students are using the terms *hypotenuse* and *leg* correctly. • Make sure that students are describing the relationship between the squares attached to each side of the triangle. • Have students verbalize what they found in the table and through class discussion, and then use the ideas provided as a springboard to move on to Examples 1 and 2.
Assessment *for* Learning	
Example 1 Have students do the Show You Know related to Example 1.	• Encourage students to verbalize their thinking. • You may wish to have students work with a partner. • It may be helpful for some students to draw the squares on grid paper and label the measurements. They can then count to determine each area. You may wish to hand out **Master 8 Centimetre Grid Paper**. • Ensure that students are placing the right angle symbol opposite the longest side (hypotenuse) of the triangle.
Example 2 Have students do the Show You Know related to Example 2.	• Encourage students to verbalize their thinking. • You may wish to have students work with a partner. • Again, some students may wish to draw the three squares on grid paper and then count to determine each area.



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The symbol ≠ means "is not equal to."

Solution

a) $5 \times 5 = 25$ The area is 25 cm². $7 \times 7 = 49$ The area is 49 cm². $9 \times 9 = 81$ The area is 81 cm².

b) Calculate the sum of the areas of the two smaller squares.
$25 + 49 = 74$
The sum of the areas is 74 cm². The sum does not equal the area of the large square. 74 cm² ≠ 81 cm²
The triangle is not a right triangle.

Show You Know

A triangle has side lengths of 12 cm, 16 cm, and 20 cm.
a) What are the areas of the three squares that can be drawn on the sides of the triangle?
b) Is the triangle a right triangle? Explain.

Key Ideas

- In a right triangle, the sum of the areas of the squares attached to the legs equals the area of the square attached to the hypotenuse.
- The Pythagorean relationship states that in a right triangle with sides s, t, and v, where side v is the hypotenuse, $v^2 = s^2 + t^2$.

Communicate the Ideas

1. Describe, using words and symbols, the relationship among the areas of the three squares shown.
2. A triangle has side lengths of 7 cm, 11 cm, and 15 cm. Explain how you can determine whether or not it is a right triangle.
3. For the triangle shown, Kendra wrote the Pythagorean relationship as $r^2 = p^2 + q^2$. Is she correct? Explain.

3.2 Exploring the Pythagorean Relationship • MHR 91

Key Ideas

The Pythagorean relationship states that the square of the hypotenuse is equal to the sum of the squares of the two legs of any right triangle. Conversely, any triangle that has this relationship among the squares of its sides is a right angle triangle.

Communicate the Ideas

In this section, students formalize their understanding of the Pythagorean relationship. In #1, students state the Pythagorean relationship, using their own words and using symbols. In #2, they explain that the triangle is a non-example of a right triangle by determining and then adding the squares of the two smaller sides, and comparing the sum to the square of the hypotenuse. In #3, students identify the common error of mistaking a leg of the triangle for the hypotenuse when using the Pythagorean relationship.

Meeting Student Needs

- Have students work in pairs or small groups to create a poster or a dramatization explaining the Pythagorean relationship in their own words.

Common Errors

- Some students may use the Pythagorean relationship incorrectly by adding the sides of the right triangle instead of the squares attached to the sides of the right triangle.

R_x Encourage students to draw full diagrams of the right triangle with the squares when they work with the Pythagorean relationship to reinforce that the relationship involves the squares of the side lengths.

WWW Web Link

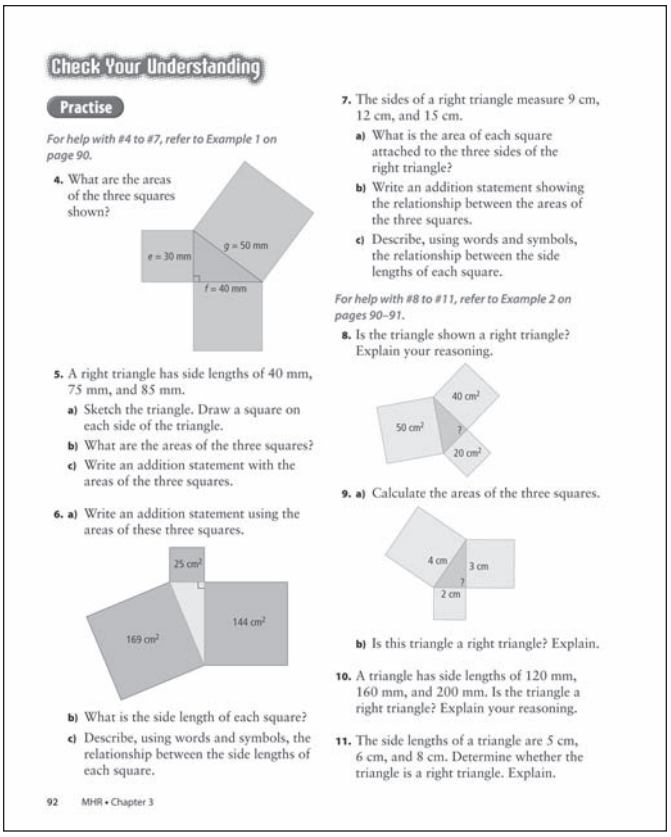
Students can explore the Pythagorean relationship by making connections between the area of each square attached to the sides of a right triangle. Go to www.mathlinks8.ca and follow the links.

Answers

Communicate the Ideas

1. The sum of the areas of the two smaller squares is equal to the area of the large square: $64 + 225 = 289$.
2. Determine the areas of the squares attached to the three sides by squaring the value of each side length: $7^2 = 49$, $11^2 = 121$, and $15^2 = 225$. Then, determine if the sum of the two smaller squares is equal to the value of the large square. Since $49 + 121$ is equal to 170 and not 225, the triangle is not a right triangle.
3. No. Explanations may vary. Example: The equation must be written so that the square of the hypotenuse equals the sum of the squares of the two legs. The equation should be $p^2 = q^2 + r^2$.

Assessment	Supporting Learning
Assessment *as* Learning	
Communicate the Ideas Have all students complete #1 to #3.	• Encourage students to verbalize their thinking. • You may wish to have students work with a partner. • Allow students to respond to #1 using written or oral form, or using symbols or diagrams. • In #2, ensure that students are squaring each side properly and writing a sum statement involving the three squares. This is an opportunity to check that students can square numbers with and without technology. • For students who need help with #3, you may wish to have them first work backwards by copying the diagram and labelling it with the appropriate variables from the Pythagorean relationship presented in the question. Then, have them explain how to write the Pythagorean relationship correctly, based on the triangle in the student resource.



Check Your Understanding

Practise

For help with #4 to #7, refer to Example 1 on page 90.

4. What are the areas of the three squares shown?

5. A right triangle has side lengths of 40 mm, 75 mm, and 85 mm.
- **a)** Sketch the triangle. Draw a square on each side of the triangle.
- **b)** What are the areas of the three squares?
- **c)** Write an addition statement with the areas of the three squares.

6. a) Write an addition statement using the areas of these three squares.

- **b)** What is the side length of each square?
- **c)** Describe, using words and symbols, the relationship between the side lengths of each square.

7. The sides of a right triangle measure 9 cm, 12 cm, and 15 cm.
- **a)** What is the area of each square attached to the three sides of the right triangle?
- **b)** Write an addition statement showing the relationship between the areas of the three squares.
- **c)** Describe, using words and symbols, the relationship between the side lengths of each square.

For help with #8 to #11, refer to Example 2 on pages 90–91.

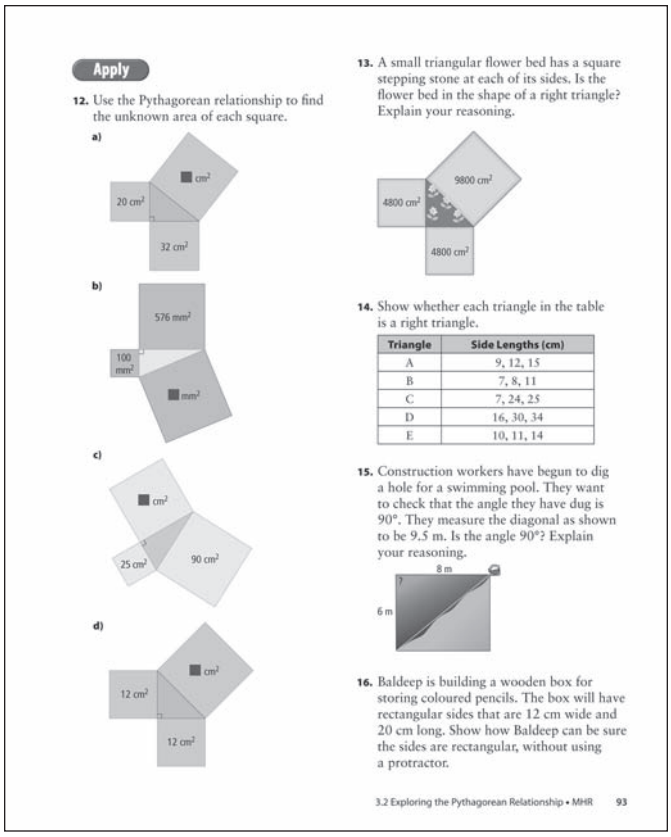
8. Is the triangle shown a right triangle? Explain your reasoning.

9. a) Calculate the areas of the three squares.

- **b)** Is this triangle a right triangle? Explain.

10. A triangle has side lengths of 120 mm, 160 mm, and 200 mm. Is the triangle a right triangle? Explain your reasoning.

11. The side lengths of a triangle are 5 cm, 6 cm, and 8 cm. Determine whether the triangle is a right triangle. Explain.



Apply

12. Use the Pythagorean relationship to find the unknown area of each square.

a)

b)

c)

d)

13. A small triangular flower bed has a square stepping stone at each of its sides. Is the flower bed in the shape of a right triangle? Explain your reasoning.

14. Show whether each triangle in the table is a right triangle.

Triangle	Side Lengths (cm)
A	9, 12, 15
B	7, 8, 11
C	7, 24, 25
D	16, 30, 34
E	10, 11, 14

15. Construction workers have begun to dig a hole for a swimming pool. They want to check that the angle they have dug is 90°. They measure the diagonal as shown to be 9.5 m. Is the angle 90°? Explain your reasoning.

16. Baldeep is building a wooden box for storing coloured pencils. The box will have rectangular sides that are 12 cm wide and 20 cm long. Show how Baldeep can be sure the sides are rectangular, without using a protractor.

Check Your Understanding

Practise

Observe whether students can answer #5 and #9. If they can complete these questions without difficulty, they likely have a good grasp of the concepts. For #9, verify that students check the sum of the squares of the two shorter sides against the square of the hypotenuse.

Apply

For #15, you may need to discuss with students the definition of the word *diagonal*.

For #16, encourage students to draw a diagram to help them with their thinking. They need to recognize that a rectangle must have right angles at its vertices.

Extend

For #20, students might want to check whether other shapes attached to the sides of a right triangle will exhibit the same relationship among their areas. For example, ask them whether the areas of equilateral triangles attached to each side would have the same relationship. Students may find it interesting that they do.

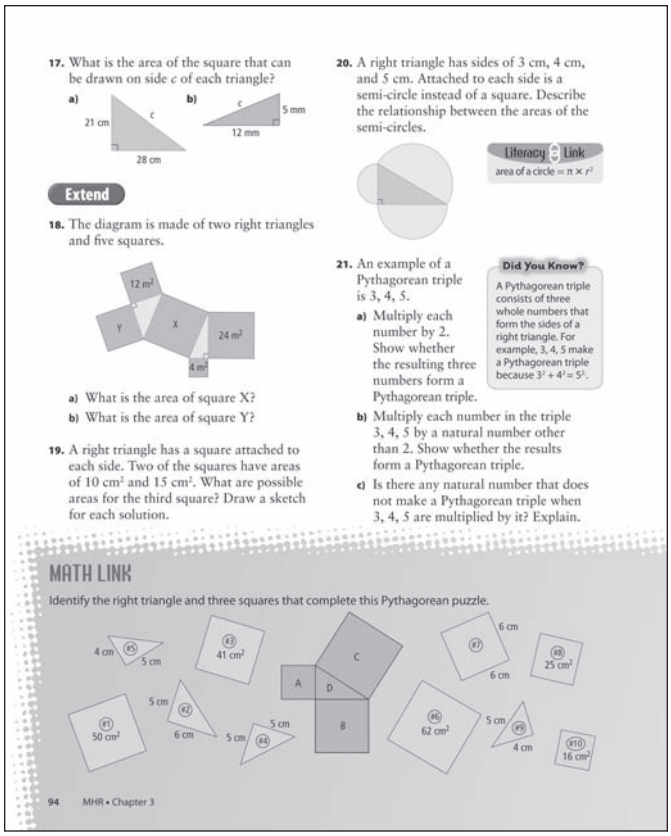
Literacy Link For #20, refer students to the Literacy Link that reminds them about the formula for the area of a circle. The chapters on geometry appear later in the student resource, so students may not have used the formula since grade 7.

Math Link

This exercise is a logic puzzle for checking students' understanding of the Pythagorean relationship. Emphasize that the drawings are not to scale. Students need to focus on the numbers and not on the relative sizes of the pieces.

Meeting Student Needs

- The Check Your Understanding questions are intended to be very visual, which will benefit visual learners.
- Some students may wish to use virtual manipulatives to sketch the diagram for #5.
- Provide **BLM 3–10 Section 3.2 Extra Practice** to students who would benefit from more practice.



17. What is the area of the square that can be drawn on side c of each triangle?

a) 21 cm, 28 cm, c

b) 12 mm, 5 mm, c

Extend

18. The diagram is made of two right triangles and five squares.

12 m², 24 m², 4 m², X, Y

a) What is the area of square X?

b) What is the area of square Y?

19. A right triangle has a square attached to each side. Two of the squares have areas of 10 cm² and 15 cm². What are possible areas for the third square? Draw a sketch for each solution.

20. A right triangle has sides of 3 cm, 4 cm, and 5 cm. Attached to each side is a semi-circle instead of a square. Describe the relationship between the areas of the semi-circles.

Literacy Link: area of a circle = $\pi \times r^2$

21. An example of a Pythagorean triple is 3, 4, 5.

a) Multiply each number by 2. Show whether the resulting three numbers form a Pythagorean triple.

b) Multiply each number in the triple 3, 4, 5 by a natural number other than 2. Show whether the results form a Pythagorean triple.

c) Is there any natural number that does not make a Pythagorean triple when 3, 4, 5 are multiplied by it? Explain.

Did You Know?

A Pythagorean triple consists of three whole numbers that form the sides of a right triangle. For example, 3, 4, 5 make a Pythagorean triple because $3^2 + 4^2 = 5^2$.

MATH LINK

Identify the right triangle and three squares that complete this Pythagorean puzzle.

4 cm, 5 cm, #5; 41 cm², #3; A, B, C, D; 6 cm, 6 cm, #7; 25 cm², #8; 5 cm, 6 cm, #2; 50 cm², #1; 5 cm, 5 cm, #4; 62 cm², #6; 5 cm, 4 cm, #9; 16 cm², #10

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Gifted and Enrichment

- Once they have completed #21, you may wish to have students find out more information about what Pythagorean triplets are. Invite them to share their findings with the class.

Answers

Math Link

Triangle #5, Square #10, Square #8, and Square #3
Area of square attached to one leg of Triangle #5: $4^2 = 16$ cm^2
Area of square attached to other leg of Triangle #5: $5^2 = 25$ cm^2
Area of square attached to hypotenuse of Triangle #5: $16 + 25 = 41$ cm^2
Area of Square #10 = 16 cm^2
Area of Square #8 = 25 cm^2
Area of Square #3 = 41 cm^2

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #5, #6, #8, #10, and #12a) and c). Students who have no problems with these questions can go on to the remaining Apply questions.	• Some students may be better able to complete #5 and #6 by referring to their answers to #2 in Communicate the Ideas. Drawing a diagram may also be of assistance. Students should try #7 before going on with the other questions. • For #8, have students identify the names of the sides to see if this helps link their understanding to a solution. Then, have students try #9. Since #8 involves the key concepts of the lesson, students should explore this question fully before continuing. • When working on #12, students may find it helpful to refer back to #6. If necessary, provide some coaching. Use #12b) to coach them through the question, and then have them complete #12d). They can then continue with the rest of the questions.
Math Link The Math Link on page 94 is intended to help students work toward the chapter problem wrap-up titled Wrap It Up! on page 115.	• It is not essential for students to complete the Math Link in order to do the Wrap It Up! at the end of the chapter. However, since the Math Link will allow students to clarify their thinking in a unique and creative way, it is recommended. • It may help some students to refer back to the Explore the Math. • Students who need help getting started could use **BLM 3–11 Section 3.2 Math Link**, which provides scaffolding.
Assessment *as* Learning	
Math Learning Log Give students the following information: A right triangle has side lengths of 9 mm, 40 mm, and 41 mm. Have students record their responses to the following prompts: • Sketch a picture of this triangle. • Sketch the squares that extend off each of the three sides. • Explain the Pythagorean relationship, using your sketch.	• It might be a useful exercise for students to construct this triangle accurately using a ruler and protractor. • Encourage students to use the What I Need to Work On tab of their chapter Foldable to note what they continue to have difficulties with.