Exploring Tessellations with Regular and Irregular Polygons

12.1

***MathLinks 8*, pages 446–451**

Suggested Timing
50–60 minutes

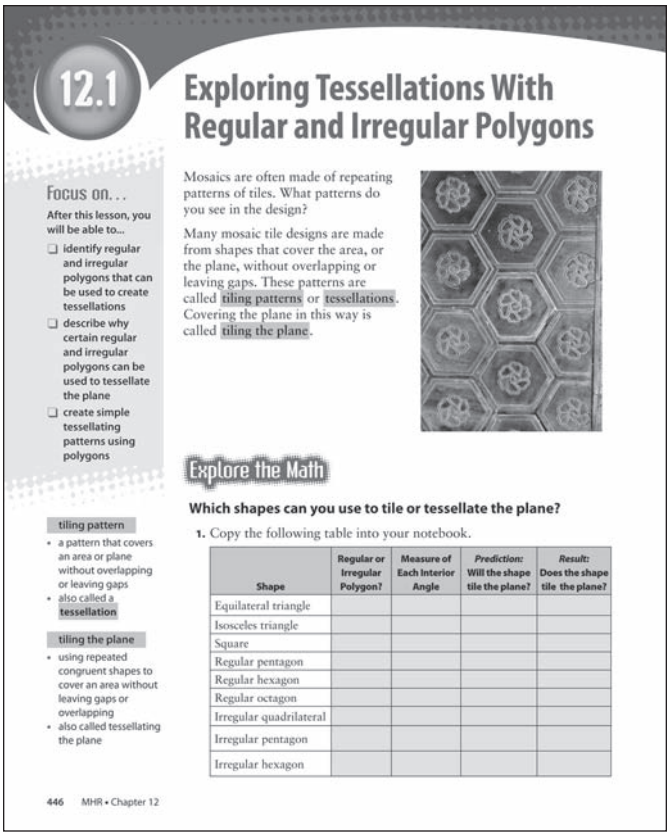
Materials
- set of pattern blocks or cardboard cutouts of pattern block shapes
- protractor
- cardboard cutouts of an isosceles triangle, a square, a hexagon, a regular pentagon, and a regular octagon
- cardboard
- scissors
- ruler
- transparent shapes
- tracing paper (optional)

Blackline Masters
Master 2 Two Stars and One Wish
Master 9 0.5 Centimetre Grid Paper
Master 7 Isometric Dot Paper
BLM 12–3 Chapter 12 Warm-Up
BLM 12–5 Section 12.1 Explore the Math
BLM 12–6 Irregular Polygons
BLM 12–7 Section 12.1 Extra Practice
BLM 12–8 Section 12.1 Math Link
BLM 12–9 Design Templates

Mathematical Processes
- ☑ Communication (C)
- ☑ Connections (CN)
- ☐ Mental Mathematics and Estimation (ME)
- ☑ Problem Solving (PS)
- ☐ Reasoning (R)
- ☐ Technology (T)
- ☑ Visualization (V)

Specific Outcomes
SS6 Demonstrate an understanding of tessellation by:
- explaining the properties of shapes that make tessellating possible
- creating tessellations
- identifying tessellations in the environment.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	1, 2, 4, 6, Math Link
Typical	1, 2, 4, 6, 8–10, Math Link
Extension/Enrichment	1, 2, 7, 8, 11–14



12.1

Exploring Tessellations With Regular and Irregular Polygons

Focus on...
After this lesson, you will be able to...
- identify regular and irregular polygons that can be used to create tessellations
- describe why certain regular and irregular polygons can be used to tessellate the plane
- create simple tessellating patterns using polygons

Mosaics are often made of repeating patterns of tiles. What patterns do you see in the design?

Many mosaic tile designs are made from shapes that cover the area, or the plane, without overlapping or leaving gaps. These patterns are called tiling patterns or tessellations. Covering the plane in this way is called tiling the plane.

Explore the Math

Which shapes can you use to tile or tessellate the plane?

1. Copy the following table into your notebook.

Shape	Regular or Irregular Polygon?	Measure of Each Interior Angle	Prediction: Will the shape tile the plane?	Result: Does the shape tile the plane?
Equilateral triangle				
Isosceles triangle				
Square				
Regular pentagon				
Regular hexagon				
Regular octagon				
Irregular quadrilateral				
Irregular pentagon				
Irregular hexagon				

tiling pattern
- a pattern that covers an area or plane without overlapping or leaving gaps
- also called a tessellation

tiling the plane
- using repeated congruent shapes to cover an area without leaving gaps or overlapping
- also called tessellating the plane

446 MHR • Chapter 12

Planning Notes

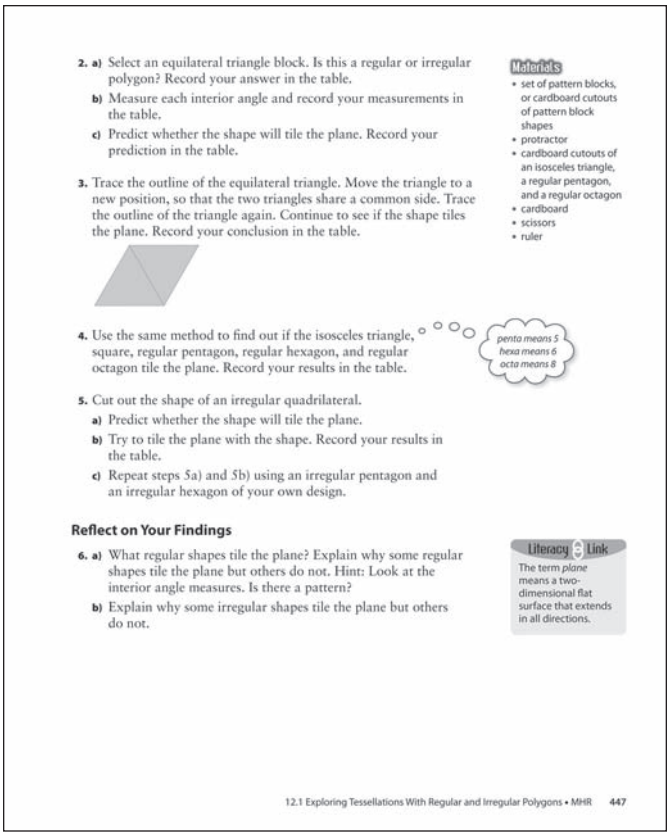
Have students complete the warm-up questions on **BLM 12–3 Chapter 12 Warm-Up** to reinforce material learned in previous sections.

As a class, read and discuss the information about mosaics in the student resource. Have students determine individually the shapes and patterns in the picture. Then, have students compare their answers.

Using transparent shapes on the overhead, show students examples of patterns where tiles effectively tile the plane and where they do not. Ensure that students are able to correctly identify and measure interior angles of polygons.

Explore the Math

In this exploration, students predict which shapes will tessellate the plane and then manipulate the shapes to test their prediction. Then, students consider what characteristics allow shapes to tessellate a plane.



2. a) Select an equilateral triangle block. Is this a regular or irregular polygon? Record your answer in the table.
b) Measure each interior angle and record your measurements in the table.
c) Predict whether the shape will tile the plane. Record your prediction in the table.

3. Trace the outline of the equilateral triangle. Move the triangle to a new position, so that the two triangles share a common side. Trace the outline of the triangle again. Continue to see if the shape tiles the plane. Record your conclusion in the table.

Materials
- set of pattern blocks, or cardboard cutouts of pattern block shapes
- protractor
- cardboard cutouts of an isosceles triangle, a regular pentagon, and a regular octagon
- cardboard
- scissors
- ruler

4. Use the same method to find out if the isosceles triangle, square, regular pentagon, regular hexagon, and regular octagon tile the plane. Record your results in the table.

penta means 5
hexa means 6
octa means 8

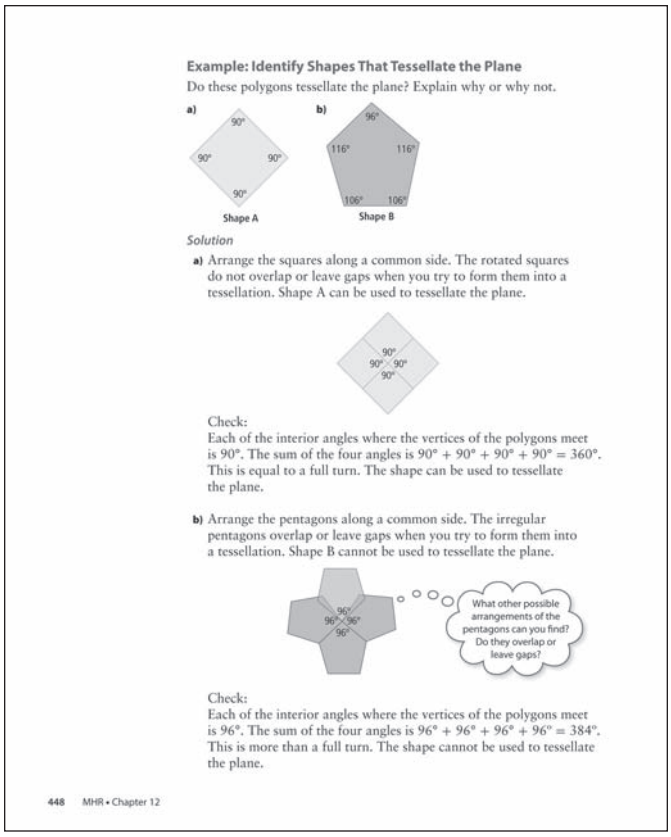
5. Cut out the shape of an irregular quadrilateral.
a) Predict whether the shape will tile the plane.
b) Try to tile the plane with the shape. Record your results in the table.
c) Repeat steps 5a) and 5b) using an irregular pentagon and an irregular hexagon of your own design.

Reflect on Your Findings

6. a) What regular shapes tile the plane? Explain why some regular shapes tile the plane but others do not. Hint: Look at the interior angle measures. Is there a pattern?
b) Explain why some irregular shapes tile the plane but others do not.

Literacy Link
The term *plane* means a two-dimensional flat surface that extends in all directions.

12.1 Exploring Tessellations With Regular and Irregular Polygons • MHR 447



Example: Identify Shapes That Tessellate the Plane

Do these polygons tessellate the plane? Explain why or why not.

a) b)

Shape A Shape B

Solution

a) Arrange the squares along a common side. The rotated squares do not overlap or leave gaps when you try to form them into a tessellation. Shape A can be used to tessellate the plane.

Check:
Each of the interior angles where the vertices of the polygons meet is 90°. The sum of the four angles is 90° + 90° + 90° + 90° = 360°. This is equal to a full turn. The shape can be used to tessellate the plane.

b) Arrange the pentagons along a common side. The irregular pentagons overlap or leave gaps when you try to form them into a tessellation. Shape B cannot be used to tessellate the plane.

What other possible arrangements of the pentagons can you find? Do they overlap or leave gaps?

Check:
Each of the interior angles where the vertices of the polygons meet is 96°. The sum of the four angles is 96° + 96° + 96° + 96° = 384°. This is more than a full turn. The shape cannot be used to tessellate the plane.

448 MHR • Chapter 12

Point out to students the thought bubble on page 447. Brainstorm several words that use the prefixes *penta*, *hexa*, and *octa*. Point out that the prefix *octa* is sometimes changed to end with an *o*, such as in *octopus* and *octogenarian*.

Method 1 Have students work on the exploration in pairs and discuss their answers. Give each pair of students a supply of pattern blocks or cardboard cutouts of pattern block shapes, so that they can manipulate the shapes concretely.

Use transparent shapes and the overhead to demonstrate the method for determining whether an equilateral triangle is regular or irregular and for measuring interior angles. Then, have students complete #3 independently. For #4, demonstrate the method for an isosceles triangle and a square, then have students use the same method to find out if the regular pentagon, regular hexagon, and regular octagon tile the plane.

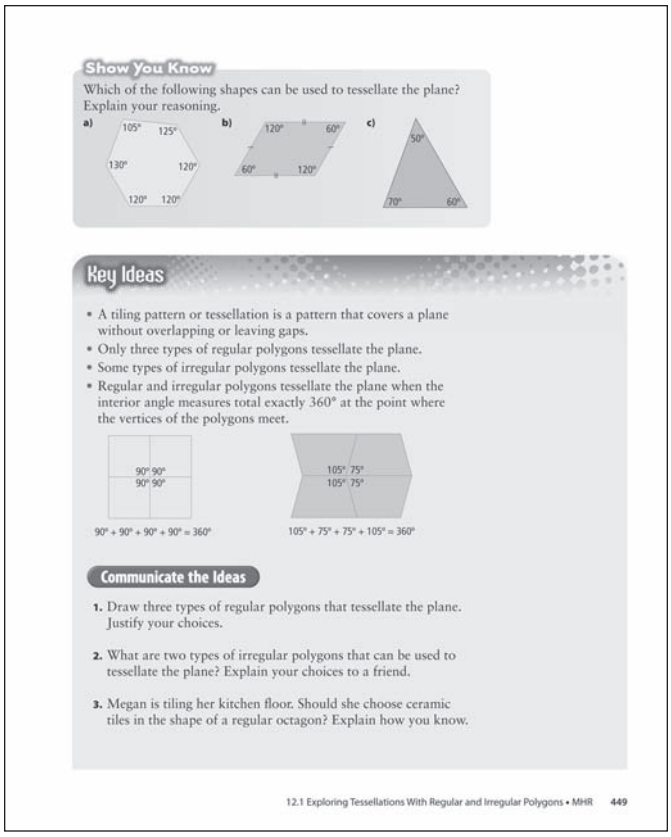
Encourage students to answer #6 independently and then compare answers, noting any differences.

Method 2 Work with shapes on an overhead to explore which shapes tile the plane and which do not. Do the first two shapes as a class, and then have students do the second two on their own. Discuss the class findings.

Check the class findings using a regular hexagon and a regular octagon. Discuss whether these shapes support or refute the opening conclusions suggested by the class. In groups, have students test alternative conclusions using the last three shapes.

Discuss with students their final conclusions and how they can be used to answer #6.

Literacy Link Discuss with students the Literacy Link on page 447. Assist them in understanding the term *plane*.



Show You Know

Which of the following shapes can be used to tessellate the plane? Explain your reasoning.

a) 105° 125° 130° 120° 120° 120°

b) 120° 60° 60° 120°

c) 50° 70° 60°

Key Ideas

- A tiling pattern or tessellation is a pattern that covers a plane without overlapping or leaving gaps.
- Only three types of regular polygons tessellate the plane.
- Some types of irregular polygons tessellate the plane.
- Regular and irregular polygons tessellate the plane when the interior angle measures total exactly 360° at the point where the vertices of the polygons meet.

90° 90° 90° 90°

90° + 90° + 90° + 90° = 360°

105° 75° 105° 75°

105° + 75° + 75° + 105° = 360°

Communicate the Ideas

1. Draw three types of regular polygons that tessellate the plane. Justify your choices.
2. What are two types of irregular polygons that can be used to tessellate the plane? Explain your choices to a friend.
3. Megan is tiling her kitchen floor. Should she choose ceramic tiles in the shape of a regular octagon? Explain how you know.

12.1 Exploring Tessellations With Regular and Irregular Polygons • MHR 449

Example

The Example shows students how to determine whether the shapes tessellate the plane. Remind students that the shape can tile the plane when the surface is totally covered with no overlap or gaps. Ensure that students correctly identify the interior angles of polygons and use this information to show why a shape can or cannot tile the plane.

You may wish to have students try to tessellate b) using other angles of the pentagon. Discuss why it will not tessellate along these angles either.

Meeting Student Needs

- Have students work with a partner or in a small group so that the work can be shared.
- Allow students to use concrete materials, such as pre-cut shapes or manufactured tiles. This will facilitate their explorations of the concepts related to tiling the plane.
- You may wish to provide students with **BLM 12–5 Section 12.1 Explore the Math** instead of having them copy the table for #1.
- You may wish to provide students with **BLM 12–6 Irregular Polygons** for students to cut out for #5.
- You may wish to invite someone with a special interest in quilts to talk to the class about quilt patterns. Check with your local community to see which cultural groups are involved in quilting and how to approach members of these groups. Have students identify tessellations in the quilt patterns they are shown.

ELL

- Ensure students understand the following terms: *interior angle*, *outline*, *common side*, *tile the plane*, *pentagon*, *isosceles triangle*, and *irregular quadrilateral*.

Gifted and Enrichment

- Have students research additional words that use the prefixes *penta*, *hexa*, and *octa*.

Common Errors

- Some students may assume the interior angles of all polygons equal 360°. This will result in students using tiles that overlap or leave gaps when tiling the plane.

R_x Remind students that they need to measure the interior angles where the polygons meet during the tessellation. These interior angles must add to exactly 360°. If they add to more or less, the shape will not tessellate. You may wish to have them study solutions a) and b) on page 448.

Answers

Explore the Math

1.-5.

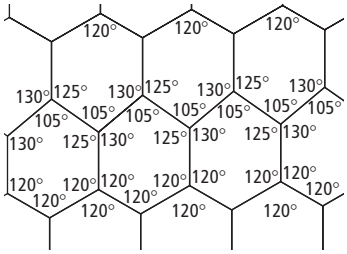
Shape	Regular or Irregular Polygon?	Measure of Each Interior Angle	*Prediction:* Will the shape tile the plane?	*Result:* Does the shape tile the plane?
Equilateral triangle	regular	60°	Answers may vary.	yes
Isosceles triangle	irregular	Answers may vary.	Answers may vary.	yes
Square	regular	90°	Answers may vary.	yes
Regular pentagon	regular	108°	Answers may vary.	no
Regular hexagon	regular	120°	Answers may vary.	yes
Regular octagon	regular	135°	Answers may vary.	no
Irregular quadrilateral	irregular	Answers may vary.	Answers may vary.	yes
Irregular pentagon	irregular	Answers may vary.	Answers may vary.	some may
Irregular hexagon	irregular	Answers may vary.	Answers may vary.	some may

6. a) Answers may vary. Example: An equilateral triangle, a square, and a regular hexagon can cover a large area without overlapping or leaving gaps. So, they can tile the plane. A regular pentagon and a regular octagon leave spaces so they cannot tile the plane. A regular polygon tiles the plane when the interior angle measures total exactly 360° at the point where the vertices of the polygon meet.

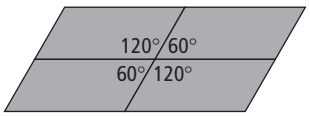
b) Answers may vary. Example: An isosceles triangle and an irregular quadrilateral tile the plane while an irregular pentagon and an irregular hexagon may not. An irregular quadrilateral tiles the plane because the interior angle measures total exactly 360° at the point where the vertices of the quadrilateral meet. Two congruent isosceles triangles form a parallelogram, so any isosceles triangle tiles the plane.

Show You Know: Example

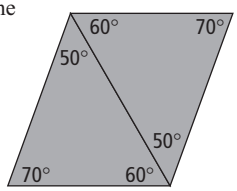
a) Yes. When copies of the irregular hexagon are put together along the common sides, the interior angle measures total exactly 360° at the point where the three different vertices meet.
$105° + 125° + 130° = 360°$
$120° + 120° + 120° = 360°$



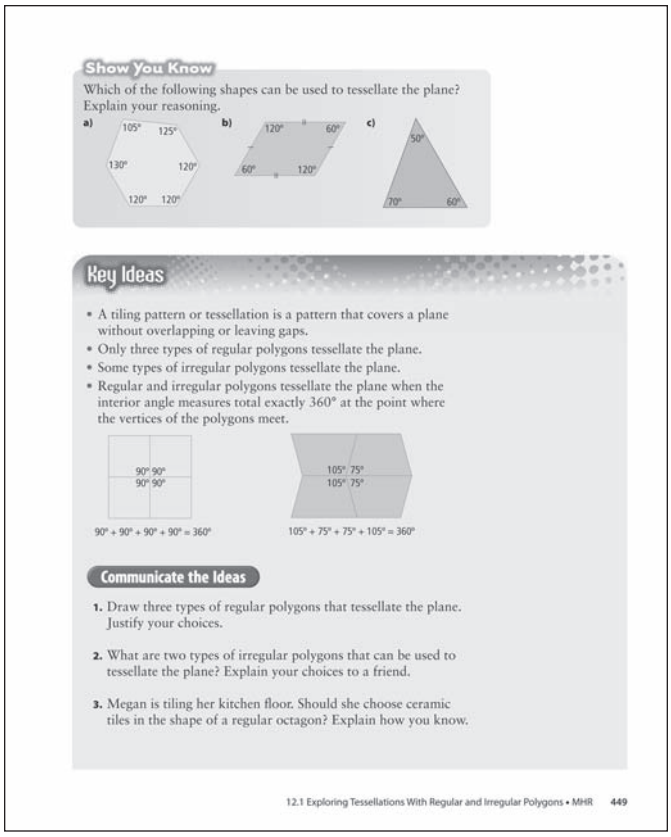
b) Yes. The interior angle measures total exactly 360° at the point where the four vertices of the parallelograms meet.
$120° + 60° + 120° + 60° = 360°$



c) Yes. Any triangle tessellates the plane because two congruent triangles form a parallelogram.



Assessment	Supporting Learning
Assessment *as* Learning	
Reflect on Your Findings Listen as students discuss which shapes can be used to tile the plane. Particular attention should be paid to the concept of the sum of interior angles adding up to 360° where the tiles meet. Try to have students generalize the conclusion about their findings.	• Reinforce the difference between tiling the plane and not tiling the plane. • Encourage and coach students to check the measures of the interior angles to determine if they add up to 360° where they meet. • Ask students to use a square and/or a 30-60-90 triangle as well as a shape that does not tile the plane to demonstrate the visual and algebraic differences between tiling and non-tiling shapes. • Clarify the terms *regular* and *irregular* polygons for student understanding.
Assessment *for* Learning	
Example Have students do the Show You Know related to the Example.	• Encourage students to verbalize their thinking. • You may wish to have students work with a partner. • Remind students to check the measures of the interior angles to determine if they add up to 360° where they meet. • You may wish to provide additional examples and non-examples of regular and irregular polygons that tile the plane. Use an overhead with different shapes in different colours of acetate to provide an effective visual for showing which shapes can or cannot tile the plane. • Some students may benefit from using tracing paper to trace the shape of the tile and use it to transform the tiling piece. • Some students may benefit from using tiling manipulatives.



Show You Know

Which of the following shapes can be used to tessellate the plane? Explain your reasoning.

a) 105° 125° 130° 120° 120° 120°

b) 120° 60° 60° 120°

c) 50° 70° 60°

Key Ideas

- A tiling pattern or tessellation is a pattern that covers a plane without overlapping or leaving gaps.
- Only three types of regular polygons tessellate the plane.
- Some types of irregular polygons tessellate the plane.
- Regular and irregular polygons tessellate the plane when the interior angle measures total exactly 360° at the point where the vertices of the polygons meet.

90° 90° 90° 90°

90° + 90° + 90° + 90° = 360°

105° 75° 105° 75°

105° + 75° + 75° + 105° = 360°

Communicate the Ideas

1. Draw three types of regular polygons that tessellate the plane. Justify your choices.
2. What are two types of irregular polygons that can be used to tessellate the plane? Explain your choices to a friend.
3. Megan is tiling her kitchen floor. Should she choose ceramic tiles in the shape of a regular octagon? Explain how you know.

12.1 Exploring Tessellations With Regular and Irregular Polygons • MHR 449

Key Ideas

Have students prepare their own list of regular polygons that tile the plane, along with notes explaining why certain irregular polygons can also be used to tile the plane. This list could be added to their chapter Foldable.

Communicate the Ideas

You may wish to have students complete the questions in groups and discuss their answers. They can then write their own justifications. Use students' responses to assess their understanding of which polygons can be used to tile the plane and why. Have a brief discussion to summarize the major points after students have attempted the questions.

Common Errors

- Students may incorrectly measure the interior angles where the shapes will meet when trying to determine if a polygon can be used to tile the plane.

R_x Encourage students to practise measuring angles and compare their results with a partner.

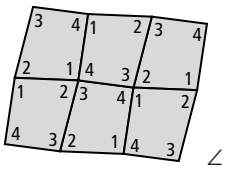
Answers

Communicate the Ideas

1. 

Answers may vary. Example: A triangle can always tessellate the plane, and the sum of the four corners of a square or three of the vertices of a regular hexagon is 360° at the point where the vertices of the polygons meet.

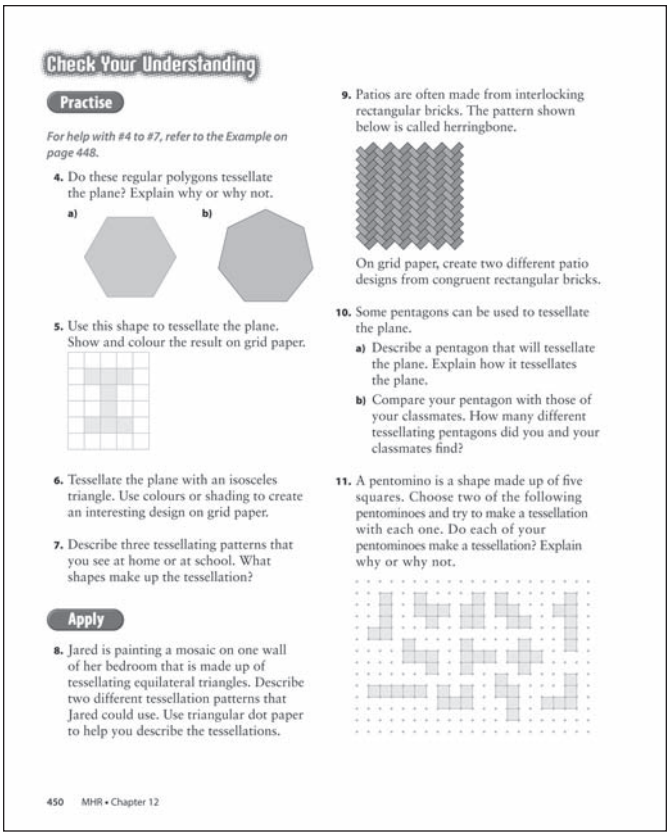
2. Answers may vary. Example: Isosceles triangles and irregular quadrilaterals can be used to tessellate the plane. A triangle can always tessellate the plane, and any irregular quadrilateral can be used to tessellate the plane because the sum of the interior angle measures is 360° at the point where the vertices of the quadrilateral meet.



$\angle 1 + \angle 2 + \angle 3 + \angle 4 = 360°$

3. Answers may vary. Example: No, Megan should not choose regular octagons. She would need tiles of another shape to cover the floor since a regular octagon cannot tile the plane by itself.

Assessment	Supporting Learning
Assessment *as* Learning	
Communicate the Ideas Have all students complete #1 and #2.	• Work with students to develop criteria for judging each answer. For example, criteria for #1 might include – correctly measures interior angles of polygon – determines the sum of the angles is 360° – checks for overlaps or gaps when tiling the plane with a polygon • Make a list of tiling and non-tiling polygons. • Students may benefit from looking back at the activity and the conclusions in the Explore the Math as well as the Example. • Have students use tracing paper to trace the shape of the tile and to transform the tiling piece. • Have students continue to use tiling manipulatives as needed. • Have students exchange their work with a classmate to see if they find any mistakes or can suggest improvements. You may wish to provide students with **Master 2 Two Stars and One Wish** for recording their feedback.



Check Your Understanding

Practise

For help with #4 to #7, refer to the Example on page 448.

4. Do these regular polygons tessellate the plane? Explain why or why not.

a) b)

5. Use this shape to tessellate the plane. Show and colour the result on grid paper.

6. Tessellate the plane with an isosceles triangle. Use colours or shading to create an interesting design on grid paper.

7. Describe three tessellating patterns that you see at home or at school. What shapes make up the tessellation?

Apply

8. Jared is painting a mosaic on one wall of her bedroom that is made up of tessellating equilateral triangles. Describe two different tessellation patterns that Jared could use. Use triangular dot paper to help you describe the tessellations.

9. Patios are often made from interlocking rectangular bricks. The pattern shown below is called herringbone.

On grid paper, create two different patio designs from congruent rectangular bricks.

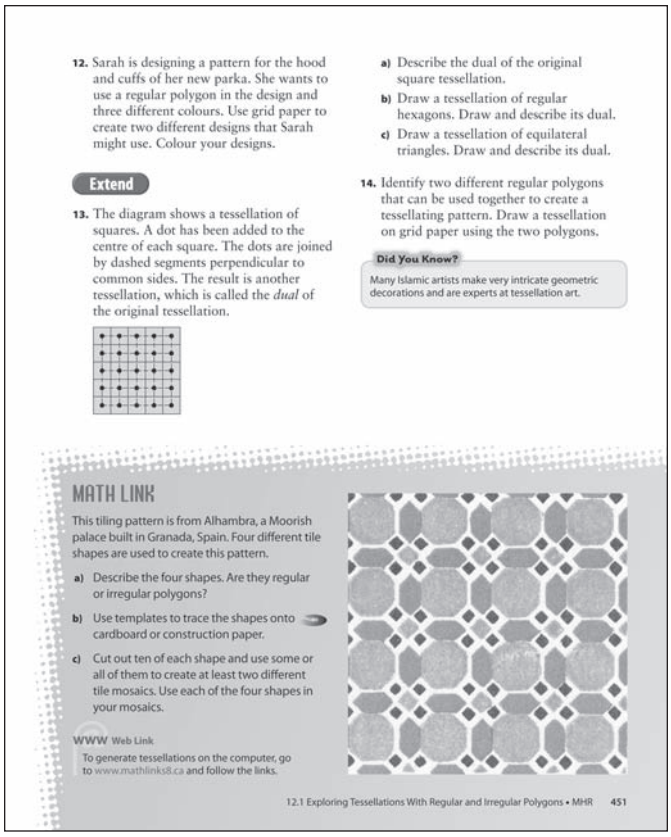
10. Some pentagons can be used to tessellate the plane.

a) Describe a pentagon that will tessellate the plane. Explain how it tessellates the plane.

b) Compare your pentagon with those of your classmates. How many different tessellating pentagons did you and your classmates find?

11. A pentomino is a shape made up of five squares. Choose two of the following pentominoes and try to make a tessellation with each one. Do each of your pentominoes make a tessellation? Explain why or why not.

450 MHR • Chapter 12



12. Sarah is designing a pattern for the hood and cuffs of her new parka. She wants to use a regular polygon in the design and three different colours. Use grid paper to create two different designs that Sarah might use. Colour your designs.

Extend

13. The diagram shows a tessellation of squares. A dot has been added to the centre of each square. The dots are joined by dashed segments perpendicular to common sides. The result is another tessellation, which is called the *dual* of the original tessellation.

a) Describe the dual of the original square tessellation.

b) Draw a tessellation of regular hexagons. Draw and describe its dual.

c) Draw a tessellation of equilateral triangles. Draw and describe its dual.

14. Identify two different regular polygons that can be used together to create a tessellating pattern. Draw a tessellation on grid paper using the two polygons.

Did You Know?

Many Islamic artists make very intricate geometric decorations and are experts at tessellation art.

MATH LINK

This tiling pattern is from Alhambra, a Moorish palace built in Granada, Spain. Four different tile shapes are used to create this pattern.

a) Describe the four shapes. Are they regular or irregular polygons?

b) Use templates to trace the shapes onto cardboard or construction paper.

c) Cut out ten of each shape and use some or all of them to create at least two different tile mosaics. Use each of the four shapes in your mosaics.

WWW Web Link

To generate tessellations on the computer, go to www.mathlinks8.ca and follow the links.

12.1 Exploring Tessellations With Regular and Irregular Polygons • MHR 451

Check Your Understanding

Practise

For #4, ensure that students justify their answers by accurately measuring the interior angles and using this information to report the sum of angles where the tiles could meet. You may wish to have students work in small groups for other Practise questions.

Apply

Students need to be able to apply their knowledge of regular and irregular polygons and understanding of the characteristics of tessellations to answer these problems. Allow students to use manipulatives as needed. Provide students with **Master 7 Isometric Dot Paper** to complete #8. Provide students with **Master 9 0.5 Centimetre Grid Paper** to complete #9 and #12.

Extend

The Extend problems require students to use greater creativity when visualizing, designing, and creating patterns. Again, allow students to use manipulatives as needed. Provide students with **Master 9 0.5 Centimetre Grid Paper** to complete #13 and #14.

Math Link

In this Math Link, students explore how different tile shapes are used to create a pattern. Make sure that students correctly identify the four shapes and categorize the shapes as regular or irregular polygons. You may need to explain that the tiles have been put together using grout, which has created the irregular white area around each tile. Encourage students to use all the shapes to create at least two different mosaics.

Meeting Student Needs

- For #4 and #5, some students may find it helpful to trace the shapes and experiment to see if they tessellate the plane. They can use this experimentation to confirm their calculations.
- For #9, some students may be unfamiliar with patios. You may wish to provide a picture of a patio to help these students visualize the question.
- For the Math Link, some students may benefit from using **BLM 12–9 Design Templates** to trace the shapes.
- Provide **BLM 12–7 Section 12.1 Extra Practice** to students who would benefit from more practice.

Common Errors

- Some students may be unsure of how to apply their knowledge to real-world situations.

R_x Provide students with more examples of tessellations and have them practise determining the shapes used to make up the pattern and whether the pattern is a tessellation.

Answers

Math Link

a) The shapes are an octagon, a hexagon, a small square, and a larger square. The octagon and the two squares are regular polygons, and the hexagon is an irregular polygon.

b, c) Answers may vary. Look for the following:
- The designs should include each of the four shapes.
- There should be no spaces between the mosaic tiles and no overlaps.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise Have students do #4 and #6. Students who have no problems with these questions can go on to the Apply questions.	• Provide additional coaching with the Example to students who need help with #4. Have them verbalize their understanding. • Encourage students to use manipulative pattern blocks to try out different shapes. • Help students who need assistance with #6 to recall the meaning of *isosceles*. They may benefit from being coached through the drawing of one pattern before moving on independently.
Math Link The Math Link on page 451 is intended to help students work toward the chapter problem wrap-up titled Wrap It Up! on page 469.	• Have students complete the Math Link, since they will use these basic skills when they design and construct their own mosaic in the Wrap It Up! • Some students will benefit from the trials they attempt in this Math Link when refining their choices for the Wrap It Up! • Students may benefit from using **BLM 12–9 Design Templates** to trace the shapes. • To help them get started, some students may benefit from using **BLM 12–8 Section 12.1 Math Link**, which provides scaffolding for this activity.
Assessment *as* Learning	
Math Learning Log Have students complete the following statements: • The difference between a regular and an irregular polygon is … • I know a polygon can tile the plane by considering …	• Depending on students' learning style, have them provide oral or written answers. • Encourage students to use diagrams of regular and irregular polygons. They could also use polygon tiles to trace rather than sketch their own polygons. • Encourage students to use the What I Need to Work On section of their chapter Foldable to note what they continue to have difficulties with.